Some Mathematical & Physical Aspects of Unity Root Matrix Theory 3x3

Richard J. Miller
richard@microscitech.com

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This is a short presentation of some of the notable mathematical and physical aspects of the 3x3 formulation (URM3) of Unity Root Matrix Theory (URMT) [1].

The work has subsequently been extended to an $n \times n$ matrix formulation [2], retaining all of the URM3 features presented herein. Complex extensions to the n-dimensional formulation of URMT complete the mathematical foundations upon which further physical aspects are derived.

Acronyms

DCE Dynamical Conservation Equation
GR General Relativity
QM Quantum Mechanics
STR Special Theory of Relativity
URM3 Unity Root Matrix Theory, 3x3.
URMT Unity Root Matrix Theory

References - see the last page

More **free PDFs**, [3] to [5], on URMT are available at http://www.urmt.org/

Contents

1 Dynamical Equations

Every description of a system has a set of equations describing its state and evolution.

Here are three innocuous linear equations, they couple three objects (x, y, z) via six dynamical variables P, Q, R, $\overline{P}, \overline{Q}, \overline{R}$ with a single, invariant eigenvalue C

$$Cx = Ry + \overline{Q}z \,,$$
$$Cy = \overline{R}x + Pz \,,$$
$$Cz = Qx + \overline{P}y \,.$$

- The equations and variables are ternary in nature
  x3 'coordinates' x, y, z
  x3 dynamical variables P, Q, R and their conjugate forms $\overline{P}, \overline{Q}, \overline{R}$,
  real $C > 0$, (an eigenvalue), $-C$ and 0 appear later.

- The equations are tightly coupled whereby each coordinate depends on the other two, with no favoured coordinate.

- The dynamical variables have conjugacy properties, e.g. the product $P\overline{P}$ is equivalent to the squared modulus of a complex number, $|Z|^2 = ZZ^*$.
  The sum $P + \overline{P}$ is equivalent to twice the real component of a complex number, i.e. $2\,\mathrm{Re}(Z) = Z + Z^*$. Conjugacy gives the theory a Hermitian, QM nature.

- A later restriction to integers for all variables means that:

  Dynamical variables become power residues $P^n \equiv \overline{P}^n \equiv C^n \pmod{x}$, and unity roots when $C = 1$, e.g. $P\overline{P} \equiv 1 \pmod{x}$.

  Conjugate relations exist between the dynamical variables $P^{n-1} \equiv \overline{P} \pmod{x}$.

2 A Conservation Equation

No conservation law, no physics.

Defining the matrix $\mathbf{A}$, coordinate vector $\mathbf{X}_+$

$$\mathbf{A} = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix},\ \mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

then the dynamical equations in matrix form are, for eigenvalue $\lambda = C$, $C > 0$

$$\mathbf{A}\mathbf{X}_+ = C\mathbf{X}_+ .$$

Defining the Kinetic term K and Potential term V as

$$K = P\overline{P} + Q\overline{Q} + R\overline{R}\ ,\ V = \frac{(PQR + \overline{P}\overline{Q}\overline{R})}{C},$$

then, for eigenvalue C, the non-singular condition, $\det(\mathbf{A} - C\mathbf{I}) = 0$, gives the

Dynamical Conservation Equation (DCE)

$$C^2 = K + V .$$

- The eigenvalue C is invariably set to unity, i.e. $C = 1$, but its presence is retained for dimensional consistency in all equations.
- A consistent set of physical $units(P, \overline{P}, ...C) = LT^{-1}$ can be applied throughout, giving $units(P\overline{P}) = L^2T^{-2}$, i.e. velocity squared = energy $E = C^2$ (per unit mass).
- When in integers:
  The eigenvalue $C \geq 1$ implies a finite, zero-point energy $E \geq 1$.
  The cubic term $PQR + \overline{P}\overline{Q}\overline{R}$ remains divisible by C, despite appearances.

3 Invariance Transformations

What remains invariant under transformation?

Consider three **local** '$\eta, \delta, \varepsilon$' variations in the following, variational matrix $\mathbf{\Delta}$

$$\mathbf{\Delta} = \begin{pmatrix} 0 & +\eta z & -\eta y \\ -\delta z & 0 & +\delta x \\ +\varepsilon y & -\varepsilon x & 0 \end{pmatrix}.$$

By definition, this matrix transforms (annihilates) the coordinate vector $\mathbf{X}_+$ as in

$$\mathbf{\Delta X}_+ = 0 .$$

Subsequently, the dynamical equations and eigenvector $\mathbf{X}_+$ remain invariant when transforming $\mathbf{A} \to \mathbf{A} + \mathbf{\Delta}$, i.e.

$$(\mathbf{A} + \mathbf{\Delta})\mathbf{X}_+ = \mathbf{AX}_+ + \mathbf{\Delta X}_+ = \mathbf{AX}_+ = C\mathbf{X}_+ .$$

A second and third **global**, variational matrix can be obtained by setting all three local variations $\eta, \delta, \varepsilon$ equal (to within a sign).

When $\varepsilon = -\delta$, the matrix is termed a **global** Pythagoras delta variation, $\mathbf{\Delta}^{\mathbf{P}}$

$$\eta = \delta, \varepsilon = -\delta, \ \mathbf{\Delta}^P = \delta \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ -y & +x & 0 \end{pmatrix}, \text{ almost symmetric.}$$

When $\varepsilon = +\delta$, the matrix is termed a **global** skew delta variation, symbol $\mathbf{\Delta}^S$

$$\eta = \delta, \varepsilon = +\delta, \ \mathbf{\Delta}^S = \delta \begin{pmatrix} 0 & +z & -y \\ -z & 0 & +x \\ +y & -x & 0 \end{pmatrix}, \text{ skew symmetric.}$$

The latter, $\mathbf{\Delta}^S$, is not used further herein, but note that $\mathbf{\Delta}^s$ is similar to an Eulerian, infinitesimal rotation matrix, angles δx, δy and δz, rotation axis $\mathbf{X}_+$.

The global variation parameter δ is later replaced by an equivalent integer parameter m, as part of an analytic solution $\delta = -m$, $units(\delta, m) = T$, time.
These matrix transformations provide a local and global symmetry operator, which leaves $\mathbf{X}_+$ invariant to their action.

4 An Invariance Principle

Physics is nicer derived starting with a conservation law and invariance principle

Lets restart, with **the Dynamical Conservation Equation**, expanded in full

$$+C^2 = P\overline{P} + Q\overline{Q} + R\overline{R} + (PQR + \overline{P}\overline{Q}\overline{R})/C\ .$$

and an **Invariance Principle**

The dynamical equations and their solutions are invariant
to a coordinate translation in the dynamical variables.

The 'coordinate translation' is given by the local, variational matrix $\mathbf{\Delta}(\eta,\delta,\varepsilon)$, linking the dynamical variables with the coordinates x, y, z

$\mathbf{A} \to \mathbf{A} + \mathbf{\Delta} \Rightarrow$

$$P \to P + \delta x\ ,\ Q \to Q + \varepsilon y\ ,\ R \to R + \eta z \text{ raised,}$$
$$\overline{P} \to \overline{P} - \varepsilon x\ ,\ \overline{Q} \to \overline{Q} - \eta y\ ,\ \overline{R} \to \overline{R} - \delta z \text{ lowered.}$$

Applying these local $\eta, \delta, \varepsilon$ variations to the DCE gives three, separate, quadratic variational terms in $\delta\varepsilon$, $\eta\varepsilon$, $\eta\delta$. The dynamical equations in x, y, z are obtained by equating the terms to zero, for arbitrary variations.

e.g. $\delta\varepsilon$ term, $\Rightarrow 0 = -x^2 + xyR + xz\overline{Q} \ \Rightarrow x = Ry + \overline{Q}z$.

Additionally, there are three linear variational terms in $\eta, \delta, \varepsilon$ which, when equating to zero, also give the three solutions in the coordinates, two of which are independent,

e.g. the δ term, gives z in terms of x , $z(R + \overline{P}\overline{Q}) = x(\overline{P} + QR)$.
and the ε term gives y in terms of x , $y(\overline{Q} + RP) = x(P + \overline{Q}\overline{R})$.

This variational method gives both the dynamical equations and their solutions

One such form of eigenvector solution, out of a possible nine, is

$$\mathbf{X}_+ = \left(\frac{1}{(C^2 - P\overline{P})}\right)\begin{pmatrix} C^2 - P\overline{P} \\ C\overline{R} + PQ \\ CQ + \overline{R}\overline{P} \end{pmatrix}.$$

- Any zero divisors, e.g. $C^2 = P\overline{P}$, can be transformed away from zero by a $\mathbf{\Delta}$ transformation, leaving $\mathbf{X}_+$ invariant by definition.

5 Integers, Quantisation and Duals

QM works for sure, so some form of quantisation is required.

Another form of solution is, for y in terms of x,

$$y(C^2 - P\overline{P}) = (C\overline{R} + PQ)x .$$

The transition to integers is made by asserting the following co-primality criteria

$$\gcd(x, y, z) = 1 .$$

This implies that, for some integer α, the following conditions apply

$$(C^2 - P\overline{P}) = \alpha x \text{ and } (C\overline{R} + PQ) = \alpha y .$$

By similar consideration of all nine possible solutions, this implies, for some integers α, β and γ, the following relations

$$(C^2 - P\overline{P}) = \alpha x ,$$
$$(C^2 - Q\overline{Q}) = \beta y ,$$
$$(C^2 - R\overline{R}) = \gamma z .$$

These defining equations for α, β, γ are symmetric upon interchange of α, β, γ with x, y, z. As such, they are **duals** to the coordinates x, y, z and vice-versa.

$$\tilde{x} = \alpha , \ \tilde{y} = \beta , \ \tilde{z} = -\gamma ,$$
$$x = \tilde{\alpha} , \ y = \tilde{\beta} , \ z = -\tilde{\gamma} ,$$

α, β, γ is another triple, forming the co-vector $\mathbf{X}^+ = \begin{pmatrix}\alpha & \beta & \gamma\end{pmatrix}$, $\mathbf{X}^+\mathbf{A} = C\mathbf{X}^+$

- The transition to integers effects a quantisation of the theory.
  This quantisation forces a finite non-zero, zero-point energy $C^2 = 1$ and removes any chance of a singularity or infinity.
  Remember, quantisation resolved the ultra-violet catastrophe.

- Summing all three gives 'the Potential equation', another conservation equation

$$+2C^2 = \alpha x + \beta y + \gamma z - (PQR + \overline{P}\overline{Q}\overline{R})/C$$

in vector form

$$\mathbf{X}^+ \cdot \mathbf{X}_+ = 2C^2 + V$$

6 Variational Methods & Pythagoras

STR also works, so Pythagoras is a must.

Using the *Global Pythagoras delta variation* $\mathbf{\Delta^P}$, derived earlier, and applying to the DCE, gives a δ^2 and δ variational term, equated to zero for arbitrary variations

δ^2 term

$$0 = x^2 + y^2 - z^2 - xy(R + \overline{R}) + xz(Q - \overline{Q}) + yz(\overline{P} - P)\,.$$

δ term

$$\begin{aligned}0 &= x(QR + \overline{Q}\overline{R}) - y(RP + \overline{R}\overline{P}) + z(PQ - \overline{P}\overline{Q}) + \\ &x(\overline{P} + P) - y(\overline{Q} + Q) + z(\overline{R} - R)\end{aligned}\,.$$

From the δ^2 variational term, if the following **Pythagoras Conditions** arise

$$\overline{P} = P\ ,\ \overline{Q} = Q\,,\ \overline{R} = -R\,,$$

then another conservation equation emerges, i.e. the Pythagoras equation,

$$0 = x^2 + y^2 - z^2\,.$$

Applying the Pythagoras conditions to the linear δ expression gives a 3rd conservation equation, the 'delta equation'

$$0 = yQ + zR - xP\,,$$

and the DCE becomes the hyperbolic equation

$$C^2 = P^2 + Q^2 - R^2\,.$$

The Potential equation becomes

$$2C^2 = \mathbf{X}^+ \cdot \mathbf{X}_+\,.$$

- All these conservation equations are 'Diophantine equations', when in integers, and the realm of number theory.

From the conservation equations, three more **scalar invariants** 0, C^2, $+2C^2$ arise to go with the three eigenvalue invariants $0, \pm C$. In fact, it is possible to also get $\pm C^2$ according to eigenvector scaling +/-1.

7 Pythagoras Continued…

Under Pythagoras Conditions:

The Potential $V = (PQR + \overline{P}\overline{Q}\overline{R})/C$ vanishes, $V = 0$

Three, symmetric eigenvalues emerge with three associated eigenvectors

$$\lambda = +C\,,\ \lambda = 0\,,\ \lambda = -C\,.$$

$$\mathbf{X}_+(\lambda = C) = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \mathbf{X}_0(\lambda = 0) = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix}, \mathbf{X}_-(\lambda = -C) = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix},$$

and their conjugates

$$\mathbf{X}^- = \begin{pmatrix} x & y & -z \end{pmatrix},\ \mathbf{X}^0 = \begin{pmatrix} P & -Q & -R \end{pmatrix},\ \mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}.$$

The dual variables α , β and γ also satisfy the Pythagoras equation

$$\alpha^2 + \beta^2 - \gamma^2 = 0\,.$$

As regards '2+1' STR, Pythagoras gives null, photon-like intervals

$$0 = x^2 + y^2 - (ct)^2\,,$$

which, in unified notation ($x = \overline{x}$, $y = \overline{y}$, $z = ct$, $\overline{z} = c\overline{t} = -z$), is nicer written as

$$0 = x\overline{x} + y\overline{y} + z\overline{z}\,.$$

By defining the transformation matrix '$\mathbf{T}$ Operator' (Minkowski metric +1 signature)

$$\mathbf{T} = \mathbf{T}^T = \mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

the conjugate vectors are related to their standard counterparts $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$

$$\mathbf{X}^- = (\mathbf{T}\mathbf{X}_+)^T\,,\ \mathbf{X}^0 = (\mathbf{T}\mathbf{X}_0)^T\,,\ \mathbf{X}^+ = (\mathbf{T}\mathbf{X}_-)^T$$

8 An Analytic Solution

An analytic solution is great, but it brings determinacy with it.

Under Pythagoras conditions, this is a completely solved problem with an analytic solution for all variables, parameterised by three (yes, three more), arbitrary integers k, l and m (see further).

$$x = 2kl\,,\ y = (l^2 - k^2)\,,\ z = (l^2 + k^2)\,.$$

The dual variables α, β, γ and dynamical variables P, Q, R are obtained by solving a linear Diophantine equation in unknown integers s and t, given k and l.

$$+C = ks - lt\,.$$

This introduces some indeterminacy in an otherwise, deterministic solution.

For particular solutions s' and t', general solutions s and t, parameterised by integer m, thought of as an evolution parameter (invariably time but could be length), the general solution is

$$s = s' + ml\,,\ t = t' + mk\,,$$

$$P = -(ks + lt)\,,\ Q = (ls - kt)\,,\ R = -(ls + kt)\,,$$

$$\alpha = -2st\,,\ \beta = (t^2 - s^2)\,,\ \gamma = (t^2 + s^2)\,.$$

The parameter m is the same as $-\delta$ in the Pythagoras transformation Δ^{P}.

- The parameter m is analogous to a winding number. For example, the dynamical variable P transforms from its initial value P' as

$$P \to P' + \delta x \sim P - mx\,,$$

i.e.

$$P \equiv P' (\operatorname{mod} x)\,.$$

Thus, the evolutionary parameter m acts as a quotient, winding P around a loop of circumference x, m times.

9 Physical Interpretation

There's abstract mathematics, and there's the real world

Physical quantities such as energy per unit mass (C^2), velocity (dynamical variables P, Q, R and eigenvalue C), and time m, have already been mentioned.

To formalise, the following physical associations can be justifiably made and give a consistent, physical identity to the work

$$m \cong \text{time},$$
$$\mathbf{X}_+,\ x, y, z \cong \text{acceleration},$$
$$\mathbf{X}_{m0},\ P, Q, R, C \cong \text{velocity},$$
$$\mathbf{X}_{m-},\ \alpha, \beta, \gamma \cong \text{position}.$$

- This is one, very useful physical association, parameter m could also be length, mass or other.

The justification for this comes from the fact that, by differentiating the evolution equations (see further) for the eigenvectors $\mathbf{X}_+$, $\mathbf{X}_{m0}$ and $\mathbf{X}_{m-}$, the eigenvectors span zero, first and second order derivatives, to within a constant factor ± 2. This constant can be scaled-out as eigenvectors are arbitrary to within a scale factor.

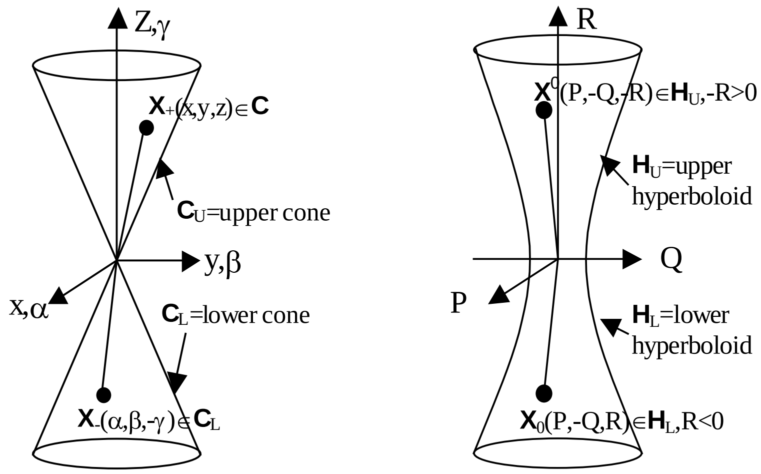
So, for example, if parameter m is associated with time, then $\mathbf{X}_{m-}$ is a position quantity (derivative of $\mathbf{X}_{m0}$), $\mathbf{X}_{m0}$ is a velocity quantity (derivative of $\mathbf{X}_+$) and $\mathbf{X}_+$ is a constant acceleration quantity (with zero derivative).

- Most importantly, this hints that calculus lurks naturally underneath. In fact, the $\mathbf{A}$ matrix and variational matrices $\mathbf{\Delta}$ can also be used as derivative, proportional, and integral operators under a unified scheme.

10 Geometry

A geometric interpretation is nice, and nicer when it looks like a page out of a relativity text.

Since $\mathbf{X}_+$ and $\mathbf{X}_-$ are Pythagorean triples, their geometry is that of a discrete cone and, since the DCE is a hyperboloid, the geometry of $\mathbf{X}_0$ is that of a discrete hyperboloid



- The union of all points in the cone $\mathbf{C}$ and the hyperboloid $\mathbf{H}$ forms a discrete lattice $\mathbf{L}$.

- As all points on the cones are Pythagorean triples, the cone's 'surface' is a zero-Potential surface since, under Pythagoras conditions, $V = 0$.

  The cone surface is therefore also one of constant energy ($= C^2$ by the DCE).

- The cones have no tip at the origin - a (0,0,0) solution is not algebraically possible for any non-zero eigenvalue, i.e. no singularity at any point in the lattice, and also, therefore, at any time; see further. Likewise, the hyperbola has a finite, non-zero radius of C at the origin.

11 Evolution

Static, time-independent solutions are a good start but, ultimately, a theory needs time evolution or equivalent.

From the analytic solution, the eigenvector $\mathbf{X}_+$ (x, y, z) is completely characterised by parameters k and l.

With k and l fixed, the eigenvectors $\mathbf{X}_0$ $(P, -Q, R)$ and $\mathbf{X}_-$ $(\alpha, \beta, -\gamma)$ are solely characterised by parameter m and evolve wrt $\mathbf{X}_+$ as

$$\mathbf{X}_{m+} = \mathbf{X}'_+(m = 0) = \mathbf{X}_+ \text{ static,}$$

$$\mathbf{X}_{m0} = -m\mathbf{X}_+ + \mathbf{X}'_0(m = 0),$$

$$\mathbf{X}_{m-} = -m^2\mathbf{X}_+ + 2m\mathbf{X}'_0(m = 0) + \mathbf{X}'_-(m = 0).$$

$\mathbf{X}_+$ is static and represents a single point in $\mathbf{C}_U$, $z > 0$ (convention).
$\mathbf{X}_{m0}$ traces out an evolving path, parameter m, on the hyperboloid $\mathbf{H}$ ($\mathbf{X}_{m0}$).
$\mathbf{X}_{m-}$ traces out an evolving path, parameter m, on the cone $\mathbf{C}_L$, $-\gamma < 0$.

- For each point on the $\mathbf{X}_+$ cone $\mathbf{C}_U$, there is an entire cone $\mathbf{C}_L$ associated with it for $\mathbf{X}_{m-}$, and a hyperboloid $\mathbf{H}$ for $\mathbf{X}_{m0}$, i.e. each $\mathbf{X}_+$ is associated with an infinite (countable) set of $\mathbf{X}_-$ and $\mathbf{X}_0$.

- Every 'point' ((x, y, z), $(P, -Q, R)$, $(\alpha, \beta, -\gamma)$) in the lattice has the same set of invariants $0, \pm C$, $\pm C^2$, $+2C^2$ arising from the conservation equations.

The increment $\delta\mathbf{X}_{m-}$ in $\mathbf{X}_{m-}$, $|m| >> 0$, approximates a scalar multiple $-2m$ of $\mathbf{X}_+$

$$\delta\mathbf{X}_{m-} \approx -2m\mathbf{X}_+, \ |m| >> 0,$$

and, since each $\mathbf{X}_+$ is a Pythagorean triple with zero norm, the evolving path of $\mathbf{X}_{m-}$ ever closer approximates a null geodesic as m grows larger.

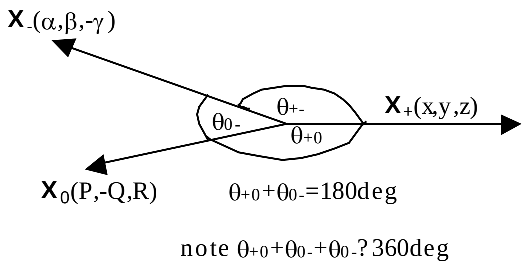
Furthermore, since every point on the cone is at zero potential, constant energy $= C^2$, there is no Kinetic/Potential energy interchange and, thus, no force acting upon it.

Therefore, $\mathbf{X}_{m-}$ effectively moves in free-fall, with constant energy, on a null geodesic in $\mathbf{C}_L$ - think mass less photon. Curiously, $\mathbf{C}_L$ is attached to $\mathbf{X}_+$, itself physically associated with a constant acceleration, when $units(m) = T$, $units(C^2) = E$.

12 Non-trivial Geometry

GR works, so something more than a globally flat, Euclidean geometry, is required.

The eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ ($\cong \mathbf{X}_{m0}$) and $\mathbf{X}_-$ ($\cong \mathbf{X}_{m-}$) are linearly independent, but are far from orthonormal, i.e. they are oblique and each of non-unit length.



Defining the **flatness parameter** ω as the ratio of the eigenvalue C to the dynamical variable R, for $R \neq 0$

$$\omega = C / R,\ R \neq 0 .$$

and, since R is parameterised by the evolution parameter m, as in $R = R' - mz$, then, for large m, and to first order in $1/m$, this approximates to

$$\omega \approx \left(\frac{-C}{z}\right)\frac{1}{m},\ |m| >> 1 .$$

Denoting the angle between $\mathbf{X}_+$ and $\mathbf{X}_-$ by θ_{+-}, for small ω, then

$$\cos(\theta_{+-}) \approx -1 + \omega^2,\ |\omega| << 1,$$

and the axes flatten out, $\cos(\theta_{+-}) \approx -1$, as evolution progresses (large m) and $\mathbf{X}_-$ evolves to become anti-parallel -180 to $\mathbf{X}_+$

- For $m = 0$ (time zero), R and z are always finite, non-zero and the flatness is well defined and finite. Conversely, the curvature (next) is also finite, non-zero.

- The evolutionary parameter m can be length, e.g. arc length, as well as time or, indeed, other.

13 Non-trivial Geometry continued…

An inverse square law is always welcome.

Defining **Curvature** κ, as the rate of change of angle θ_{+-} with respect to the evolution parameter m, i.e.,

$$\kappa = \delta\theta_{+-} / \delta m,$$

and approximating for large evolutionary times m (small flatness ω), gives the curvature as follows

$$\kappa \approx \left(\frac{\sqrt{2}C}{z}\right)\frac{1}{m^2}, \quad |m| >> 1.$$

This curvature for θ_{+-}, with respect to m, is an inverse square law *.
*** CAUTION** m here is physically associated with time (not distance), when speaking in terms of energy for the DCE and velocity for the dynamical variables.

The same relation, barring a factor, applies to angles θ_{+0} and θ_{0-} respectively.

The curvature is proportional to the eigenvalue C, which is effectively a free parameter for tuning, noting $C^2 \propto E$, i.e. energy per unit mass (when m is time)

Non-trivial geometry:

evolving curvature,

always finite and non-zero,

with no singularity.

14 Symmetry Breaking

Too much symmetry, not enough complexity, something has to break.

Under Pythagoras conditions, the global transformation $\mathbf{\Delta^P}$ preserves the zero Potential and gives force-free, constant energy, null geodesic trajectories for $\mathbf{X}_-$ on the lower cone.

However, any one of the three local variations, whilst still leaving $\mathbf{X}_+$ and the total energy C^2 invariant, destroys the zero Potential and triggers an interchange of kinetic and Potential energy along both the $\mathbf{X}_-$ and $\mathbf{X}_0$ trajectories (parameterised by the local variation itself).

The $\mathbf{X}_-$ vector is no longer a Pythag triple and the lower cone is destroyed. Neither is the $\mathbf{X}_0$ vector preserved and the hyperbola is also destroyed. Barring the invariant $\mathbf{X}_+$ vector, the symmetry relating to the other two eigenvectors is completely broken.

In other words, the local variation has broken the global symmetry and, by exchange of potential and kinetic energy, induced a force.

There are three possible local variations in the 3x3 theory and so, in principle, three possible forces can be induced. This sounds good (too good) but are they all fundamentally the same? It is not so simple now...

15 Duality

Very much in vogue, nice if you can get it.

As mentioned earlier, but now under Pythagoras conditions, all equations are symmetric upon interchange of the dual variables α, β, γ with x, y, z.

The divisibility factors α, β, γ are therefore dual to the coordinates x, y, z and, consequently, $\mathbf{X}_{m-}$ is also the dual eigenvector to $\mathbf{X}_{+}$, i.e. $\mathbf{X}_{m-} = \tilde{\mathbf{X}}_{+}$ or $\mathbf{X}_{+} = \tilde{\mathbf{X}}_{m-}$.

From the evolution equations, $\mathbf{X}_{m-}$ evolves with respect to $\mathbf{X}_{+}$ for large m, as

$$\mathbf{X}_{m-} \approx -m^2 \mathbf{X}_{+}, \ |m| >> 0$$

Thus, the vector $\mathbf{X}_{m-}$ tends to look like $\mathbf{X}_{+}$, scaled by m^2 and, barring scale, the two worlds $\mathbf{X}_{m-}$ and $\mathbf{X}_{+}$ look the same.

So, by studying the world of $\mathbf{X}_{+}$, then simply rescaling by m^2, gives the world of $\mathbf{X}_{m-}$ and vice versa, i.e. you can work in the $\mathbf{X}_{+}$ world, or its dual $\mathbf{X}_{-}$ (but not both simultaneously).

In terms of the null-cone sets $\mathbf{C}_L, \mathbf{C}_U$, this represents a duality between the small and large-scale geometry expressed as $\mathbf{C}_L = \tilde{\mathbf{C}}_U$ and $\mathbf{C}_U = \tilde{\mathbf{C}}_L$.

The middle ground (macroscopic world) is considered to be that of the eigenvector $\mathbf{X}_{m0}$ residing in the disjoint, hyperbolic set $\mathbf{H}$.

Relative to $\mathbf{X}_{m0}$, the microscopic region is $\mathbf{X}_{+}$ and the large scale region that of $\mathbf{X}_{m-}$ so, for large m,

$$\begin{aligned} &\mathbf{X}_{+}, \ \text{micro} \\ \mathbf{X}_{m0} &\approx -m\mathbf{X}_{+}, \ \text{macro} \\ \mathbf{X}_{m-} &\approx -m^2\mathbf{X}_{+}, \ \text{large} \end{aligned}$$

Dividing throughout by m, and when viewed with respect to $\mathbf{X}_{m0}$, for large m

$$\mathbf{X}_{+} \to \frac{1}{m}\mathbf{X}_{+}, \ \mathbf{X}_{m-} \to m\mathbf{X}_{+}$$

and

$\mathbf{X}_{m0}$ sees an $m, \dfrac{1}{m}$ duality between the microscopic and the very large.

16 And beyond...

Invariably there are always limitations, open questions, and extensions required.

STR is a given, so the preceding 2+1 formulation is not sufficient; four-vectors, e.g. (x, y, z, ct), and non-zero intervals $c\tau$, are required. Some more complexity and more dynamical variables might be nice too (for all the numerous particles)?

The theory can be extended to 4D, whilst retaining the existing properties, by embedding 3x3 in a 4x4 formulation. Additionally, this naturally adds full conjugacy ($\bar{x}, \bar{y}, \bar{z}$), $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ and non-trivial 4-vectors $\mathbf{X}_-$ (further below)

$$\mathbf{A} = \begin{pmatrix} 0 & \bar{x} & \bar{y} & \bar{z} \\ x & 0 & R & \overline{Q} \\ y & \overline{R} & 0 & P \\ z & Q & \overline{P} & 0 \end{pmatrix}, \ \mathbf{X}_+ = \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}, \ \overline{\mathbf{X}}_+ = \mathbf{X}^- = \begin{pmatrix} 0 & \bar{x} & \bar{y} & \bar{z} \end{pmatrix},$$

$$\overline{\mathbf{X}}_+ \cdot \mathbf{X}_+ = 0, \ \mathbf{A}\mathbf{X}_+ = C\mathbf{X}_+, \ \overline{\mathbf{X}}_+\mathbf{A} = -C\overline{\mathbf{X}}_+ \text{ (Hermitian).}$$

Solutions can be lifted, retaining an invariant, 3-element $\mathbf{X}_+$, but now obtaining a non-invariant, 4-element $\mathbf{X}_-$

$$\mathbf{X}_- = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}, \ 0 = a^2 + b^2 + c^2 - d^2$$

This still gives a zero interval for $\mathbf{X}_-$, so how about embedding 4x4 in a 5x5 to get a full 4D space-time with non-zero interval (for $\mathbf{X}_-$ only, not $\mathbf{X}_+$ when lifting), something like:

$$\mathbf{A} = \begin{pmatrix} 0 & \overline{\mathbf{X}}_3 & c\bar{t} \\ \mathbf{X}_3 & 0 & \overline{\mathbf{E}} \\ ct & \mathbf{E} & \mathbf{B}_3 \end{pmatrix}, \ \mathbf{B}_3 = \begin{pmatrix} 0 & -B^3 & B^2 \\ B^3 & 0 & -B^1 \\ -B^2 & B^1 & 0 \end{pmatrix}, \ \mathbf{E} = \begin{pmatrix} E^1 \\ E^2 \\ E^3 \end{pmatrix}, \overline{\mathbf{E}} = \left(-E^1, -E^2, -E^3\right)$$

The 4x4, 5x5 and $n \times n$ extensions to URMT are the realm of [2] and [5].

With some complex extensions, the n-dimensional formulation of URMT completes the mathematical foundations of the theory, with the next publication concentrating on the physical aspects.

17 References

[1] Unity Root Matrix Theory, Physics in Integers, R J Miller, FastPrint Publishing, 2011, ISBN 978-184426-974-7,
http://www.fast-print.net/bookshop/823/unity-root-matrix-theory-physics-in-integers

This is the foundation book on the 3x3 formulation of URMT, and is broken into six separate papers, each paper is given a specific reference #1 to #6 as follows:

[1],#1 Unity Root Matrix Theory Foundations
[1],#2 see [4], below
[1],#3 Geometric and Physical Aspects
[1],#4 Solving Unity Root Matrix Theory
[1],#5 Unifying Concepts
[1],#6 A Non-unity Eigenvalue

[2] Unity Root Matrix Theory, Higher Dimensional Extensions, R J Miller, FastPrint Publishing, 2012, ISBN 978-178035-296-1,
http://www.fast-print.net/bookshop/1007/unity-root-matrix-theory-higher-dimensional-extensions
This book is the second published work on URMT detailing the extension of the 3x3 formulation in [1] to an arbitrary number of dimensions.

[3] *Unity Root Matrix Theory, Overview.*
http://www.urmt.org/presentation_URMT_shortform.pdf R J Miller, Issue 1.1, 09/07/2011. A **free PDF** available for download.
This presentation is an overview of the six papers, published in [1].

[4] *Pythagorean Triples as Eigenvectors and Related Invariants,*
http://www.urmt.org/pythag_eigenvectors_invariants.pdf R J Miller, 2010. A **free PDF** available for download. This paper is essentially the second paper published in [1] describing some modern results on Pythagorean triples and their derivation as eigenvectors of a unity root matrix.

[5] *Unity Root Matrix Theory Compactification of an n-dimensional eigenvector space over long evolutionary timescales,*
http://www.urmt.org/urmt_dimensional_compactification.pdf R J Miller, Issue 2 May 2012. A **free PDF** available for download.
This paper is an extract from the higher dimensional extension work in [2] illustrating one of its key results, namely the apparent shrinkage of higher dimensions over long evolutionary timescales.