Pythagorean Triples as Eigenvectors & Related Invariants

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Abstract

This paper studies a special integer matrix and its eigenvectors showing that it has two distinct Pythagorean triples as eigenvectors and a third, related eigenvector, satisfying a similar, hyperbolic Diophantine equation. Every such matrix is defined to be similar with common eigenvalues -1, 0, $+1$ and it is consequently proven that every Pythagorean triple is thus fundamentally related to a matrix with these three eigenvalues. Vector, dot-product relations between the three eigenvectors and their conjugate forms produce six additional invariants relating to the matrix and its eigenvectors. The paper finishes by providing an analytic solution for all eigenvectors, matrices and equations and hence verifying all results.

Keywords. Pythagorean Triple, Eigenvector, Eigenvalue

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(1) Introduction

(1.0) Definition. A **Pythagorean triple** considered herein comprises any ordered triple (a,b,c), of integers a,b,c that satisfy the Pythagoras equation $0 = a^2 + b^2 - c^2$. This definition is to be interpreted in its loosest sense with the only condition being that $(a,b,c) \neq (0,0,0)$. In other words, a, b and c are allowed to be positive or negative integers, and a may be less than or greater than b; a can be zero in which case $|b| = |c|$ or b can be zero in which case $|a| = |c|$. Non-primitive triples are also included, i.e. those such that, for non-zero, integer factor k, if (a,b,c) is a Pythagorean triple then so too is (ka,kb,kc). Otherwise, primitive solutions are co-prime, i.e. $\gcd(a,b,c) = 1$.

Association of Pythagorean triples with matrices is not new per se and the ability, via three linear transformations, to transform the basic triple (3,4,5) into all other Pythagorean triples was first published by Berggren [1]. Such linear transformations are represented as 3x3 matrices and this fact appears to been rediscovered in various guises thereafter, see Barning [2] which explicitly provides the 3x3 matrices. For a more modern, English language source, see [3]. However, the work presented herein appears unique and not directly related to these cited references or any other related work since it entails a matrix and its eigenvectors in what is, loosely, an identity mapping of the eigenvectors but where the matrix is not the identity.

The eigenvectors are those of the following matrix, (symbol $\mathbf{A}$), which comprises 'dynamical' integer variables P,Q,R

(1.1) $$\mathbf{A} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix}, \; P,Q,R \in \mathbb{Z}, \; (P,Q,R) \neq (0,0,0).$$

It is noted the matrix is not symmetric, skew-symmetric or invertible (it has zero determinant), all of which are common forms of matrix for study. Nevertheless, by imposition of a single constraint on its elements, the eigenvalues and eigenvectors of $\mathbf{A}$ prove to be of interest in the study of Pythagoras's theorem.

Before proceeding to the details, the results presented in this paper actually stem from simplifications made to a more general theory, and as yet unpublished, on a special form of matrix of which $\mathbf{A}$ (1.1) is a simplification.

(2) Eigenvalues and Eigenvectors of A

Defining the following vector $\mathbf{X}$ in the integer variables x,y,z

(2.1) $$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; x,y,z \in \mathbb{Z}, \; (x,y,z) \neq (0,0,0),$$

then the matrix eigenvalue equation for matrix $\mathbf{A}$ (1.1), eigenvalue λ, is

(2.2) $\mathbf{AX} = \lambda\mathbf{X}$,

and the characteristic equation is

(2.3) $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$.

Expanding (2.3) in full gives

(2.4) $-\lambda^3 + \lambda(P^2 + Q^2 - R^2) = 0$.

Imposing the following constraint in the variables P, Q, R, termed the 'dynamical conservation equation',

(2.5) $+1 = P^2 + Q^2 - R^2$,

then the characteristic equation (2.4) factors as follows

(2.6) $\lambda(\lambda - 1)(\lambda + 1) = 0$,

with three eigenvalues

(2.7) $\lambda = +1$, $\lambda = 0$, $\lambda = -1$.

To each of these eigenvalues correspond three eigenvectors, $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ satisfying their defining transformations

(2.8)
(2.8a) $\mathbf{AX}_+ = \mathbf{X}_+$,
(2.8b) $\mathbf{AX}_0 = 0$,
(2.8c) $\mathbf{AX}_- = -\mathbf{X}_-$.

Expanding equation (2.8a) for the $\mathbf{X}_+$ eigenvector, using (1.1) for $\mathbf{A}$, gives the three linear equations

(2.9)
(2.9a) $x = Ry + Qz$
(2.9b) $y = -Rx + Pz$
(2.9c) $z = Qx + Py$.

Multiplying (2.9a) by x, (2.9b) by y and (2.9c) by z, it is seen that x, y, z satisfy the Pythagoras Theorem

(2.10) $0 = x^2 + y^2 - z^2$,

and so the eigenvector $\mathbf{X}_+$, for eigenvalue $\lambda = +1$, is thus a Pythagorean triple

(2.11) $$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \; x, y, z \in \mathbb{Z}, \; (x, y, z) \neq (0,0,0).$$

Defining the eigenvector $\mathbf{X}_-$ for the eigenvalue $\lambda = -1$, by the triple of integers α, β, γ as follows, where the inverted sign on γ is intentional,

(2.12) $$\mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}, \; \alpha, \beta, \gamma \in \mathbb{Z}, \; (\alpha, \beta, \gamma) \neq (0,0,0),$$

and expanding equation (2.8c) for the $\mathbf{X}_-$ eigenvector, using (1.1) for $\mathbf{A}$, gives the three linear equations

(2.13)

(2.13a) $-\alpha = R\beta - Q\gamma$

(2.13b) $-\beta = -R\alpha - P\gamma$

(2.13c) $+\gamma = Q\alpha + P\beta$.

Just as in the $\lambda = +1$ case, multiplying (2.13a) by α, (2.13b) by β and (2.13c) by γ, it is seen that the eigenvector $\mathbf{X}_-$ also satisfies the Pythagoras Theorem.

(2.14) $$0 = \alpha^2 + \beta^2 - \gamma^2.$$

Thus the eigenvector $\mathbf{X}_-$ for the eigenvalue $\lambda = -1$ is another Pythagorean triple, distinct from $\mathbf{X}_+$ as its eigenvalue is also distinct.

It is noted that the reasoning above can also give two Pythagorean triples if the $\lambda = +1$ equations (2.9) and $\lambda = -1$ equations (2.13) use $\lambda = +C$ and $\lambda = -C$ instead, for some arbitrary, integer constant $C \geq 1$. However, this is an unnecessary complication for the purposes of this paper and its conclusions, and it is found sufficient to work with the simplest, non-zero, integer eigenvalues $\lambda = \pm 1$.

For completeness, the eigenvector $\mathbf{X}_0$ is defined here and justified later. However, simple algebraic manipulation of (2.8b) using $\mathbf{A}$ (1.1) shows this to be correct, to within a scale factor, which is sufficient for an eigenvector solution. Note that $\mathbf{X}_0$ is not a Pythagorean triple.

(2.15) $$\mathbf{X}_0 = \begin{pmatrix} +P \\ -Q \\ +R \end{pmatrix}.$$

Thus, so far, every matrix $\mathbf{A}$ (1.1) with elements P,Q,R, that satisfies constraint (2.5), has an eigenvector $\mathbf{X}_+$ for eigenvalue $\lambda=+1$ and an eigenvector $\mathbf{X}_-$ for eigenvalue $\lambda=-1$, both of which are always Pythagorean triples. Given the trace of $\mathbf{A}$ is zero then the sum of the eigenvalues is also zero and hence $\lambda=0$ is the third eigenvalue with related eigenvector $\mathbf{X}_0$ (2.15).

Having specified eigenvalues, the normal procedure would now be to obtain specific eigenvector solutions in terms of the elements P,Q,R of $\mathbf{A}$. This isn't actually of prime importance here and is deferred to later when analytic solutions are obtained. However, solving the linear equations represented by (2.9), for the specific eigenvector $\mathbf{X}_+$ (2.11), is an important and necessary step to formally defining integers α,β,γ, introduced in (2.12) for $\mathbf{X}_-$.

Algebraically solving equations (2.9) gives six direct expressions relating x, y and z, only two of which are, of course, linearly independent by the nature of the eigenvalue problem.

(2.16)

(2.16a) $\quad y(1-P^2)=(-R+PQ)x$

(2.16b) $\quad z(1-Q^2)=(P+QR)y$

(2.16c) $\quad x(1+R^2)=(Q+RP)z$

(2.16d) $\quad z(1-P^2)=(Q-RP)x$

(2.16e) $\quad x(1-Q^2)=(R+PQ)y$

(2.16f) $\quad y(1+R^2)=(P-QR)z$

Another three alternative relations can also be obtained by equating pairs of the above six.

(2.16g) $\quad z(-R+PQ)=y(Q-RP)$

(2.16h) $\quad z(R+PQ)=x(P+QR)$

(2.16i) $\quad y(Q+RP)=x(P-QR)$

All these nine forms are used further below, in section 3, when evaluating each of the nine elements of another 'residual' matrix.

It should be noted that any potential divide-by-zero or indefinite 0/0 terms in (2.16), arising due to particular values of the dynamical variables P, Q and R, can actually be transformed away by altering them in such a way that the triple (x,y,z) remains invariant. This is possible because the analytic solution, given in Appendix (A), shows them to be parameterised, see equations (A1.6c) for R, (A1.12c) for Q and (A1.15e) for P.

So far no specific condition that all variables be restricted to integers has actually been necessary, merely a presumption given the paper is about Pythagoras. However, an important restriction to integers is now made and serves to formally define the divisibility parameters α, β, γ as first used in (2.12).

(3) Co-primality Criteria and Divisibility Factors

For arbitrary integers α, β, γ, termed 'divisibility factors', equations (2.16) imply, for co-prime, primitive, integer solutions x, y, z, the following relations (3.1).

(3.1)

(3.1a) $\alpha x = (1 - P^2)$

(3.1b) $\beta y = (1 - Q^2)$

(3.1c) $\gamma z = (1 + R^2)$

Since non-primitive triples are allowed, i.e. triples (kx, ky, kz) for non-zero, integer factor k, then any common factor k can effectively be absorbed into α, β, γ, i.e. transferred from x, y, z to α, β, γ such that the triple (α, β, γ) now becomes non-primitive, i.e. $(k\alpha, k\beta, k\gamma)$.

The definitions (3.1) imply the following, additional co-primality relations for the remaining equations in (2.16),

(3.1d) $\alpha y = (-R + PQ)$

(3.1e) $\alpha z = (Q - RP)$

(3.1f) $\beta x = (R + PQ)$

(3.1g) $\beta z = (P + QR)$

(3.1h) $\gamma x = (Q + RP)$

(3.1i) $\gamma y = (P - QR)$.

In total, nine variables have now been introduced $\{x, y, z, P, Q, R, \alpha, \beta, \gamma\}$, organised as three sets of triples. This completes the set of all unknowns and the paper now proceeds to obtain the eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$, $\mathbf{X}_-$ and study the relations amongst them.

(4) Generation of Solutions as Eigenvectors

To solve for the eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ the standard method would be to solve equations (2.8) algebraically in terms of its elements, i.e. the dynamical variables P, Q and R. However, this misses a lot of new information that will effectively highlight relations between all eigenvectors and new, conjugate eigenvectors $\mathbf{X}^+$, $\mathbf{X}^0$ and $\mathbf{X}^-$, yet to be defined. The eigenvectors will, instead, be obtained by determination of a 'residual' matrix, denoted by $\mathbf{E}_+$, $\mathbf{E}_0$, $\mathbf{E}_-$ for each eigenvector $\mathbf{X}_+$, $\mathbf{X}_0$, $\mathbf{X}_-$ respectively. As a consequence of calculating a residual matrix, conjugate eigenvectors $\mathbf{X}^+$, $\mathbf{X}^0$, $\mathbf{X}^-$ are seen to arise naturally as scale factors of the eigenvectors. Note that the name 'residual' is unique to this paper as no name appears in the literature. However, the method to calculate the residual matrix is mentioned in a few texts, e.g. [4], and is also related to 'purification' or 'Richardson's purification process'. Generally it can be found in matrix-related books under the subject of numerical determination of eigenvalues and eigenvectors.

The residual method derives from a simple property of matrices and their eigenvalues known as the Cayley-Hamilton theorem [4], which basically says that a matrix satisfies its own characteristic equation.

By the Cayley-Hamilton theorem, the characteristic equation (2.6) can be re-written in terms of matrix $\mathbf{A}$ (1.1) as follows, where $\mathbf{I}$ is the identity matrix

(4.1) $\quad 0 = \mathbf{A}(\mathbf{A}-\mathbf{I})(\mathbf{A}+\mathbf{I})$.

Defining the residual matrices $\mathbf{E}_+$, $\mathbf{E}_0$, $\mathbf{E}_-$ for eigenvalues +1, 0, -1 as

(4.2)

(4.2a) $\quad \mathbf{E}_+ = (\mathbf{A}^2 + \mathbf{A})$

(4.2b) $\quad \mathbf{E}_0 = (\mathbf{A}^2 - I)$

(4.2c) $\quad \mathbf{E}_- = (\mathbf{A}^2 - \mathbf{A})$,

then (4.1) can be re-written in the following three equivalent forms (allowing for a legitimate reordering of terms)

(4.3)

(4.3a) $\quad (\mathbf{A}-\mathbf{I})\mathbf{E}_+ = 0$

(4.3b) $\quad \mathbf{A}\mathbf{E}_0 = 0$

(4.3c) $\quad (\mathbf{A}+\mathbf{I})\mathbf{E}_- = 0$.

It is now clear that $\mathbf{E}_+$, $\mathbf{E}_0$, $\mathbf{E}_-$ are termed 'residual' since each matrix is the polynomial remaining after factorisation of (4.1) by the linear factor $(\mathbf{A}-\lambda\mathbf{I})$ for $\lambda = +1, 0, -1$.

Comparing the eigenvector equation $(\mathbf{A}-\mathbf{I})\mathbf{X}_+ = 0$ (2.8a), for the $\lambda = +1$ eigenvalue, with $(\mathbf{A}-\mathbf{I})\mathbf{E}_+ = 0$ (4.3a), then it is concluded that the residual matrix $\mathbf{E}_+$ must comprise three column vectors $\mathbf{X}_+$, each equivalent, to within a scale factor, of the eigenvector $\mathbf{X}_+$. In fact, in the $\lambda = +1$ case, the scale factors will be shown to be exactly the divisibility factors α, β and γ (3.1).

The same reasoning applies to the other eigenvectors $\mathbf{X}_-$, $\mathbf{X}_0$ and residual matrices $\mathbf{E}_-$, $\mathbf{E}_0$.

Since any eigenvector is always arbitrary, to within a scale factor, the residual matrix provides three, equally valid eigenvector solutions, one in each column. All that is required to obtain the residual matrix is to evaluate its polynomial definition (4.2)

All residual matrices have an $\mathbf{A}^2$ term which, using $\mathbf{A}$ (1.1), evaluates to

(4.4) $$\mathbf{A}^2 = \begin{pmatrix} -R^2+Q^2 & +PQ & +PR \\ +PQ & -R^2+P^2 & -QR \\ -PR & +QR & P^2+Q^2 \end{pmatrix}.$$

Using the dynamical conservation equation (2.5), the leading diagonal terms can then be re-written to give

(4.5) $$\mathbf{A}^2 = \begin{pmatrix} 1-P^2 & +PQ & +PR \\ +PQ & 1-Q^2 & -QR \\ -PR & +QR & 1+R^2 \end{pmatrix}.$$

(5) Residual Matrix $\mathbf{E}_+$

Substituting for $\mathbf{A}$ (1.1) and $\mathbf{A}^2$ (4.5) into (4.2a) gives the residual matrix $\mathbf{E}_+$

(5.1) $$\mathbf{E}_+ = \begin{pmatrix} 1-P^2 & PQ+R & +PR+Q \\ PQ-R & 1-Q^2 & -QR+P \\ -PR+Q & QR+P & 1+R^2 \end{pmatrix},$$

and using the co-primality criteria (3.1) this becomes

(5.2) $$\mathbf{E}_+ = \begin{pmatrix} \alpha x & \beta x & \gamma x \\ \alpha y & \beta y & \gamma y \\ \alpha z & \beta z & \gamma z \end{pmatrix}.$$

The residual matrix $\mathbf{E}_+$ is seen to comprise three eigenvector columns with the common eigenvector $\mathbf{X}_+$ and scale factors α, β, γ, consequently $\mathbf{E}_+$ can also be written as

(5.3) $$\mathbf{E}_+ = (\alpha\mathbf{X}_+, \beta\mathbf{X}_+, \gamma\mathbf{X}_+).$$

(6) Residual Matrix $\mathbf{E}_0$

Substituting for $\mathbf{A}^2$ (4.5) into (4.2b) gives the residual matrix $\mathbf{E}_0$ as

$$\text{(6.1)} \qquad \mathbf{E}_0 = \begin{pmatrix} -PP & +QP & +RP \\ -P(-Q) & Q(-Q) & R(-Q) \\ -PR & +QR & +RR \end{pmatrix}.$$

The residual matrix $\mathbf{E}_0$ is confirmed as comprising three eigenvector columns, each with common eigenvector $\mathbf{X}_0$, as per (2.15), and scale factors $-P$, Q, and R. The residual matrix $\mathbf{E}_0$ is re-written in terms of the eigenvector $\mathbf{X}_0$ as

$$\text{(6.2)} \qquad \mathbf{E}_0 = (-P\mathbf{X}_0, Q\mathbf{X}_0, R\mathbf{X}_0).$$

Note that further in the paper, when defining eigenvector $\mathbf{X}^0$ (8.2), the scale factors $+P$, $-Q$ and $-R$ will be used in preference to $-P$, Q and R in (6.2). This change is simply a re-scaling by -1 and thus perfectly legitimate for an eigenvector, which is only unique to within an arbitrary scale factor.

(7) Residual Matrix $\mathbf{E}_-$

Substituting for $\mathbf{A}$ (1.1) and $\mathbf{A}^2$ (4.5) into (4.2c) gives the residual matrix $\mathbf{E}_-$

$$\text{(7.1)} \qquad \mathbf{E}_- = \begin{pmatrix} 1-P^2 & +PQ-R & +PR-Q \\ PQ+R & 1-Q^2 & -(QR+P) \\ -(PR+Q) & QR-P & 1+R^2 \end{pmatrix},$$

and using the co-primality criteria (3.1) this becomes

$$\text{(7.2)} \qquad \mathbf{E}_- = \begin{pmatrix} \alpha x & \alpha y & \alpha(-z) \\ \beta x & \beta y & \beta(-z) \\ (-\gamma)x & (-\gamma)y & (-\gamma)(-z) \end{pmatrix}$$

This residual matrix is seen to comprise three eigenvector columns with the common eigenvector $\mathbf{X}_-$ (2.12) and scale factors x, y and $-z$. The residual matrix $\mathbf{E}_-$ is therefore re-written as

$$\text{(7.3)} \qquad \mathbf{E}_- = (x\mathbf{X}_-, y\mathbf{X}_-, -z\mathbf{X}_-).$$

Note that since Pythagoras is an even, quadratic exponent, the minus sign in the z scale factor in (7.2) and the component $-\gamma$ in $\mathbf{X}_-$ (2.12) is immaterial since they both square to the same value as their positive counterparts; the product term γz is, of course, also invariant to a simultaneous sign change in both z and γ, i.e. $\gamma z = (-\gamma)(-z)$.

This completes the evaluation of the residual matrices $\mathbf{E}_+$, $\mathbf{E}_0$, $\mathbf{E}_-$ and consequent extraction of eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$, $\mathbf{X}_-$ for all eigenvalues $\lambda = +1, 0, -1$ respectively.

(8) Conjugate Eigenvectors

The following conjugate vectors $\mathbf{X}^+$, $\mathbf{X}^0$, $\mathbf{X}^-$ are defined by the scale factors in the residual matrices $\mathbf{E}_+$ (5.3), $\mathbf{E}_0$ (6.2) and $\mathbf{E}_-$ (7.3) respectively, as follows. Note the nomenclature is chosen so that the vector dot product relations, e.g. $\mathbf{X}_i \cdot \mathbf{X}^j \neq 0, i = j$ and $\mathbf{X}_i \cdot \mathbf{X}^j = 0, i \neq j$, are of a familiar, orthogonal form.

(8.1) $$\mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$$
(8.2) $$\mathbf{X}^0 = \begin{pmatrix} P & -Q & -R \end{pmatrix}$$
(8.3) $$\mathbf{X}^- = \begin{pmatrix} x & y & -z \end{pmatrix}$$

The sign of $\mathbf{X}^0$ (8.2) is also the reverse of the scale factor in $\mathbf{E}_0$ (6.2), and done for reasons of sign consistency upon transformation between all standard and conjugate eigenvectors, albeit this is beyond the scope of this paper. However, given $\mathbf{X}^0$ is an eigenvector, it is completely arbitrary to within a scale factor, -1 here, and so this sign reversal poses no loss of generality, noting that it makes the invariant given by (9.3), further below, a value of +1 instead of -1.

(9) Vector Products

With three standard eigenvectors and their conjugates fully defined, the following vector dot product relations between them are obtained, whereby the invariant integer value on the right is justified further below.

(9.1) $$\mathbf{X}_+ \cdot \mathbf{X}^- = x^2 + y^2 - z^2 = 0$$
(9.2) $$\mathbf{X}_- \cdot \mathbf{X}^+ = \alpha^2 + \beta^2 - \gamma^2 = 0$$
(9.3) $$\mathbf{X}_0 \cdot \mathbf{X}^0 = P^2 + Q^2 - R^2 = +1$$
(9.4) $$\mathbf{X}_+ \cdot \mathbf{X}^+ = \mathbf{X}_- \cdot \mathbf{X}^- = \alpha x + \beta y + \gamma z = +2$$
(9.5) $$\mathbf{X}_+ \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^- = xP - yQ - zR = 0$$
(9.6) $$\mathbf{X}_- \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^+ = \alpha P - \beta Q + \gamma R = 0$$

The first and second equations, (9.1) and (9.2), are the Pythagoras equation and hence zero.

The third equation (9.3) is the dynamical conservation equation (2.5) and hence equal to +1.

The fourth equation (9.4) can be verified as equal to +2 by summation of the three divisibility relations (3.1a) to (3.1c) to get

(9.7) $$3-(P^2+Q^2-R^2)=\alpha x+\beta y+\gamma z\ .$$

Using the dynamical conservation equation (2.5), the left hand term reduces to the value +2, hence (9.4).

The fifth equation (9.5) can be verified algebraically by multiplying the equation in x (2.9a) by P, y (2.9b) by Q and z (2.9c) by R and summing appropriately $(xP-yQ-zR)$ to see the sum is zero.

Justification for the sixth equation (9.6) has to wait until the analytic solution for P,Q,R and α,β,γ is obtained, which follows shortly.

For completeness, the following vector cross products are given

(9.8) $$\mathbf{X}_+\wedge\mathbf{X}_+=\mathbf{X}_0\wedge\mathbf{X}_0=\mathbf{X}_-\wedge\mathbf{X}_-=0$$
$$\mathbf{X}^+\wedge\mathbf{X}^+=\mathbf{X}^0\wedge\mathbf{X}^0=\mathbf{X}^-\wedge\mathbf{X}^-=0$$

(9.9) $$\mathbf{X}^+\wedge\mathbf{X}^0=-\mathbf{X}^0\wedge\mathbf{X}^+=\mathbf{X}_-$$
$$\mathbf{X}_+\wedge\mathbf{X}_0=-\mathbf{X}_0\wedge\mathbf{X}_+=\mathbf{X}^-$$

(9.10) $$\mathbf{X}^-\wedge\mathbf{X}^+=-\mathbf{X}^+\wedge\mathbf{X}^-=+2\mathbf{X}_0$$
$$\mathbf{X}_-\wedge\mathbf{X}_+=-\mathbf{X}_+\wedge\mathbf{X}_-=+2\mathbf{X}^0$$

(9.11) $$\mathbf{X}^0\wedge\mathbf{X}^-=-\mathbf{X}^-\wedge\mathbf{X}^0=\mathbf{X}_+$$
$$\mathbf{X}_0\wedge\mathbf{X}_-=-\mathbf{X}_-\wedge\mathbf{X}_0=\mathbf{X}^+\ .$$

Note that they are positive when the indices cycle in the order +1, 0, -1, and negative when in the order +1, -1, 0, and all vector product relations give identical results when the index is consistently raise/lowered throughout.

Using the dot product relations (9.1) to (9.6), the vector triple product is seen to be

(9.12) $$\mathbf{X}^+\wedge\mathbf{X}^0\cdot\mathbf{X}^-=\mathbf{X}_-\cdot\mathbf{X}^-=\mathbf{X}_+\wedge\mathbf{X}_0\cdot\mathbf{X}_-=\mathbf{X}^-\cdot\mathbf{X}_-=+2$$

Conversely, the vector triple product is -2 if the indices cycle in the -1, 0, +1 order, i.e.

(9.13) $$\mathbf{X}^-\wedge\mathbf{X}^0\cdot\mathbf{X}^+=-\mathbf{X}_+\cdot\mathbf{X}^+=\mathbf{X}_-\wedge\mathbf{X}_0\cdot\mathbf{X}_+=-\mathbf{X}^+\cdot\mathbf{X}_+=-2\ .$$

The vector triple product is another invariant, albeit a consequence of the previously assigned invariants (9.1) to (9.6) and cross product relations (9.8) to (9.11).

Lastly, it is also noted the following dyadic products give the residual matrices

(9.14) $\quad \mathbf{E}_+ = \mathbf{X}_+\mathbf{X}^+$

(9.15) $\quad \mathbf{E}_0 = -\mathbf{X}_0\mathbf{X}^0$

(9.16) $\quad \mathbf{E}_- = \mathbf{X}_-\mathbf{X}^-$.

Thus, all three familiar vector products, i.e. the dot, cross and dyadic products, give meaningful results.

(10) An Analytic Pythagoras Solution

Given that Pythagoras has a standard analytic solution for all triples it is, perhaps, not surprising that all variables and equations can be solved with similar, integer parameterisations. In total there are nine variables separated into three triples (P,Q,R), (x,y,z) and (α,β,γ). A complete solution is obtained when all nine unknowns $\{x,y,z,P,Q,R,\alpha,\beta,\gamma\}$ are determined that satisfy the eigenvector equations (2.8).

For the sake of brevity, the algebraic details are supplied in Appendix (A), with example data supplied in Appendix (E), and only a summary of all key results is provided here. A complete summary of all key equations is given in Appendix (B) for quick reference.

As expected, the Pythagorean triple (x,y,z) is parameterised by two arbitrary integers, k and l, subject to the following conditions which allow one, but not both, of x or y to be zero. There is also a coprimality constraint, $\gcd(k,l)=1$, applied so that the congruence (A1.3a), further below, has integer solutions.

(A1.2) $\quad k,l \in \mathbb{Z},\ (k,l) \neq (0,0),\ \gcd(k,l)=1$.

The triple (x,y,z) is then given by the familiar Pythagorean parameterisation

(A1.14b) $\quad x = 2kl$

(A1.9d) $\quad y = (l^2 - k^2)$

(A1.3d) $\quad z = (l^2 + k^2)$.

Note that the parameterisation arises through the process of actually solving the divisibility factor equations (3.1a) to (3.1c), and is not an a priori assumption. Furthermore, this particular parameterisation, with x assigned even and y assigned odd, is an arbitrary choice made in the process of obtaining a solution; the choice can equally be reversed to give odd x and even y. This is an important point since it is essential to the argument, given later, that the analytic solution for x, y and z spans all Pythagorean triples as per definition (1.0), e.g. triple $(4,3,5)$ is, strictly speaking, distinct from $(3,4,5)$. See Appendix (D) for an extension of the analytic solution in Appendix (A) to all Pythagorean triples.

To solve for (P,Q,R) and (α,β,γ), two more integers s and t are introduced as solutions to the following linear Diophantine equation (A1.3a) in integers k and l

(A1.3a) $\quad +1 = ks - lt,\ s,t \in \mathbb{Z}.$

This equation is solved by standard methods [5], to obtain two particular, integer solutions s' and t', and general solutions s and t parameterised by a third, arbitrary, integer parameter m

(A1.4a) $\quad s = s' + ml$
(A1.4b) $\quad t = t' + mk$.

Thus, given a particular solution s' and t', there are now effectively three arbitrary parameters k, l and m. Integer m can be set to zero such that $s = s'$, $t = t'$ in (A1.4a) and (A1.4b). The $m = 0$ case is also referred to as a primitive solution. However, to retain full generality, and using integer parameters k, l and m and general solutions s and t, then (P,Q,R) and (α,β,γ) are obtained from the following relations

(A1.15c) $\quad P = -(ks + lt)$
(A1.9b) $\quad Q = (ls - kt)$
(A1.3b) $\quad R = -(ls + kt)$

(A1.16a) $\quad \alpha = -2st$
(A1.9c) $\quad \beta = (t^2 - s^2)$
(A1.3c) $\quad \gamma = (t^2 + s^2)$.

Looking at the above solutions for α, β and γ, it is seen they also constitute a Pythagorean triple with the standard parameterisation excepting the sign of α. However, given integer s can be negative without affecting γ or β, this is immaterial.

Note that one of α or β can legitimately be zero. If $\alpha = 0$ then $|\beta| = |\gamma|$ and, conversely, if $\beta = 0$ then $|\alpha| = |\gamma|$. Invariant (9.4) shows they cannot all be zero and so neither can α and β both simultaneously be zero. However the single case when $\alpha = 0$ is a special case, briefly described as follows. Setting $k = 1$ in (A1.14b) gives $x = 2l$, $y = (l^2 - 1)$ and $z = (l^2 + 1)$, i.e. those Pythagorean triples where x and z differ by 2. The simplest example is when $l = 2$ and the $(4,3,5)$ triple is obtained, see Appendix (C). With $k = 1$ the solution to the Diophantine equation $+1 = ks - lt$ (A1.3a) is $s = 1 + lt$. A solution is obtained when $t = 0$ and so $s = +1$, which gives $\alpha = 0$ (A1.16a), $\beta = -1$ (A1.9c) and $\gamma = +1$ (A1.3c) and thus the scale factor triple (α,β,γ) is $(0,-1,+1)$.

For the same value of k and l, and therefore the same primitive triple (x, y, z), a different value for s and t can be chosen by varying parameter m in (A1.4) and thereby obtain a different triple (α,β,γ). Setting $m=+1$, with particular solutions $t'=0$ and $s'=+1$, then (A1.4) gives $s=+3$ and $t=+1$ (using $s=1+lt$), and the (α,β,γ) triple is $(-6,-8,+10)$, i.e. twice the primitive $(3,4,5)$ triple, disregarding sign. Setting $m=2$ gives the primitive triple $(\alpha,\beta,\gamma)=(-20,-21,+29)$, and $m=4$ gives $(-72,-65,+97)$ etc. Suffice to note, an odd value for m always returns a non-primitive triple with common factor 2.

Returning to the analytic solution, the equation (9.6) can now be verified as zero by direct substitution for α,β,γ and P,Q,R (see above), and expanding in full, then all the terms are seen to cancel as follows

(A1.30b) $$(2st)(ks+lt)-(t^2-s^2)(sl-tk)-(s^2+t^2)(sl+tk)=0\,.$$

With 3 arbitrary parameters k, l, m, and six equations, this almost completes the parametric solution to the nine unknowns (P,Q,R), (x,y,z) and (α,β,γ). Nevertheless, the analytic solution for x (A1.14b), y (A1.9d) and z (A1.3d) is known not to give all Pythagorean triples, according to the definition (1.0). To all intents and purposes, the analytic solution is usually satisfactory to derive all primitive triples. However, to include every possible solution, three extra cases require consideration. The first case is where the solution generates a triple (x,y,z) and cannot thus generate the triple (y,x,z). The second case is the inability to generate all non-primitive triples. Lastly, the third case is the inability to generate a negative value for z. The arguments to show that it is possible to extend the eigenvector solutions to include all Pythagorean triples are given in Appendix (D).

(11) Discussion

The expressions for x, y and z are all functions of k and l only whilst the expressions for α, β and γ are all functions of s and t only and parameterised by arbitrary parameter m. The expressions for the dynamical variables P, Q and R are mixed functions of all four parameters l, k, s and t and can therefore be thought of as the link between the two sets of triples (x,y,z) and (α,β,γ).

The triple (x,y,z) is uniquely defined by parameters l and k but triples (P,Q,R) and (α,β,γ) are effectively both parameterised by integer m, due to their dependence on s and t. Given m is arbitrary, then it is concluded every primitive Pythagorean triple (x,y,z) is associated with an infinite set of Pythagorean triples (α,β,γ) and an infinite set of dynamical variables (P,Q,R).

The analytic solution for (x,y,z), although essentially derived from first principles herein, is just the standard, textbook Pythagorean parameterisation and gives all primitive Pythagorean triples. Arguments in Appendix (D) extend the solution to include all Pythagorean triples and therefore the eigenvector $\mathbf{X}_+$ (2.11), of matrix $\mathbf{A}$ (1.1), spans all Pythagorean triples according to definition (1.0). Hence the eigenvalue $\lambda=+1$ can be associated with every Pythagorean triple.

Once the $\mathbf{X}_+$ eigenvector is determined, as parameterised by integers k and l, the analytic solution then gives the eigenvector $\mathbf{X}_-$ (2.12) in terms of integer parameters s and t which, themselves, are dependent upon a Diophantine equation (A1.3a) in k and l. Thus, with the $\mathbf{X}_+$ eigenvector obtained first, the $\mathbf{X}_-$ eigenvector may not, and does not, span all Pythagorean triples even though its standard Pythagorean parameterisation in s and t may make it superficially appear to be the case. This is a consequence of s and t constrained to satisfy the Diophantine equation (A1.3a). Nevertheless, the process of obtaining an analytic solution could be reversed by first obtaining the $\mathbf{X}_-$ eigenvector. The parameterisation in terms of s and t would then be free to take arbitrary values and the integer parameters k and l would now be those constrained by the Diophantine equation (A1.3a). Hence, by reversing the arguments above for $\mathbf{X}_+$ to first consider $\mathbf{X}_-$ instead, the eigenvalue $\lambda = -1$ can also be associated with every Pythagorean triple. In summary, either the set of all $\mathbf{X}_+$ or the set of all $\mathbf{X}_-$ spans the primitive Pythagorean triples but not both simultaneously. Choosing one unique solution for $\mathbf{X}_+$ effectively generates an infinite subset of Pythagorean triples $\mathbf{X}_-$ and vice-versa.

Finally, given the trace of the matrix is the sum of the eigenvalues, the only other eigenvalue has to be $\lambda = 0$ with related eigenvector $\mathbf{X}_0$ (2.15). This $\mathbf{X}_0$ eigenvector has an analytic solution for all its elements (P,Q,R) such that they satisfy the dynamical conservation equation (2.5).

(12) Conclusions

It has been shown in section (2) that any matrix of the form $\mathbf{A}$ (1.1), with elements P, Q and R, subject to the constraint (2.5), has three eigenvalues $\lambda = +1,0,-1$ and three associated eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ two of which, $\mathbf{X}_+$ and $\mathbf{X}_-$, are Pythagorean triples. A complete, analytic solution has been derived, see Appendix (A), giving integer solutions for all matrix elements P, Q and R of $\mathbf{A}$, satisfying constraint (2.5), and also providing integer solutions for all eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$. The analytic solution gives eigenvectors $\mathbf{X}_+$ that span all primitive Pythagorean triples. Additional arguments in Appendix (D) extend these eigenvectors to cover every Pythagorean triple. As a consequence, every Pythagorean triple can be associated with the eigenvector $\mathbf{X}_+$ of matrix $\mathbf{A}$, for eigenvalue $\lambda = +1$, and all such matrices $\mathbf{A}$ are similar with the same three eigenvalues $\lambda = +1,0,-1$. This proves the original assertion that every Pythagorean triple is related to a matrix with these three eigenvalues.

With three eigenvectors there are a maximum of six possible, unique, vector dot product relations between them, (9.1) to (9.6), and it is shown that all six such products produce a further six integer invariants in addition to the three eigenvalues.

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Appendices

Appendix (A)
The Analytic Solution

Obtaining a solution in all nine variables $\{x, y, z, P, Q, R, \alpha, \beta, \gamma\}$ starts by solving the defining equation (3.1c) for the divisibility factor γ, reproduced below

(3.1c) $\gamma z = (1 + R^2)$.

Consider the following algebraic identity with which (3.1c) will be compared

(A1.1) $(t^2 + s^2)(l^2 + k^2) = (ks - lt)^2 + (ls + kt)^2$.

The quantities k and l are chosen as arbitrary, integer parameters subject to the following conditions, whilst s and t are integer values to be determined

(A1.2) $k, l \in \mathbb{Z}$, $(k, l) \neq (0,0)$, $\gcd(k, l) = 1$.

These conditions allow one, but not both, of x or y to be zero. Neither is there any constraint on $1 \le k < l$ such that $y > x$. The coprimality constraint $\gcd(k, l) = 1$ is applied so that the congruence (A1.3a), further below, has integer solutions.

Each of the bracketed terms in the identity (A1.1) is assigned to integers R , γ and z as in (3.1c) to give the following relations

(A1.3)
(A1.3a) $+1 = ks - lt$, $s, t \in \mathbb{Z}$.
(A1.3b) $R = -(ls + kt)$
(A1.3c) $\gamma = (t^2 + s^2)$
(A1.3d) $z = (l^2 + k^2)$.

By (A1.3d), choosing k and l immediately fixes the value for z .

The assignment of $-(ls + kt)$ to R , as opposed to $+(ls + kt)$ in (A1.3b), is explained further below when the dynamical variable Q is evaluated.

Note that the assignment of γ and z in (A1.3c) and (A1.3d) could be swapped such that $(s^2 + t^2) = z$ and $(k^2 + l^2) = \gamma$. In fact, the choice made is such that the coordinate solution (x, y, z) that emerges is purposefully chosen to represent the $\lambda = +1$ eigenvector $\mathbf{X}_+$ and $(\alpha, \beta, -\gamma)$ to represent the eigenvector $\mathbf{X}_-$ for the $\lambda = -1$ eigenvalue.

With k and l chosen, and consequently z defined by (A1.3d), the equation (A1.3a) can be solved as a linear Diophantine equation in unknown integers s and t. Using the solutions for s and t, then R and γ can be determined from (A1.3b) and (A1.3c). See a standard number theory text, e.g. [5], for solving the congruence (A1.3a).

Denoting s' and t' as particular solutions to the Diophantine equation (A1.3a) then, for arbitrary integer m, the general solution is given by

(A1.4)
(A1.4a) $\quad s = s' + ml$
(A1.4b) $\quad t = t' + mk$.

Substituting for s and t into (A1.3c) derives the general solution for γ

(A1.5) $$\gamma = s'^2 + t'^2 + 2m(ls' + kt') + m^2(k^2 + l^2).$$

The general solution for R can be obtained by substituting for s and t into (A1.3b)

(A1.6a) $$R = -(ls' + kt') - m(l^2 + k^2).$$

Denoting a particular solution for R by R', where

(A1.6b) $$R' = -(ls' + kt'),$$

and using (A1.3d) to substitute for $(l^2 + k^2)$ in terms of z in (A1.6a), the most general solution for R becomes

(A1.6c) $$R = R' - mz.$$

Denoting a particular solution for γ as γ', then (A1.3c) becomes

(A1.7a) $$\gamma' = s'^2 + t'^2.$$

Using this and (A1.3d) for z, (A1.6b) for R', then the general solution for γ (A1.5) can be re-expressed as

(A1.7b) $$\gamma = \gamma' - 2mR' + m^2 z.$$

To summarise, by choice of integers k, l, and m, the dynamical variable R, divisibility factor γ and coordinate z can all be determined by solving a single, linear Diophantine equation (A1.3a). That even one Diophantine equation is required is due to the fact that every eigenvector solution $\mathbf{X}_+$ is actually associated with a corresponding infinite set (equivalence class) of dynamical variables, here all linked by integer parameter m. The last equation (A1.6c) shows that R is arbitrary to within a multiple m of z. Likewise, every eigenvector solution $\mathbf{X}_+$ is also associated with a corresponding infinite set of divisibility factors α, β, γ; here the value of γ (A1.7b) is arbitrary according to the choice of parameter m.

By using a very similar identity to (A1.1) below, one of the other two dynamical variables, Q or P, can also be determined in a similar manner.

(A1.8) $$(t^2 - s^2)(l^2 - k^2) = (ks - lt)^2 - (ls - kt)^2$$

The question is, which of the two, Q or P, can it be related to?

Suppose (A1.8) relates to Q, then each bracketed term in (A1.8) is assigned to integers Q, β and y as follows, whereby integers k and l are as above,

(A1.9)

(A1.9a) $+1 = (ks - lt)$

(A1.9b) $Q = (ls - kt)$

(A1.9c) $\beta = (t^2 - s^2)$

(A1.9d) $y = (l^2 - k^2)$.

The identity (A1.8) then becomes

(3.1b) $$\beta y = 1 - Q^2.$$

However, if (A1.8) was related to P instead of Q then (A1.8) would look like

(3.1a) $$\alpha x = 1 - P^2,$$

and P would be assigned to the same quantity as Q, namely,

(A1.10) $$P = (ls - kt).$$

In principle, (A1.8) could be used to determine either P or Q. As might be suspected, it is largely irrelevant except, once a choice is made, the other dynamical variable cannot be derived from the same equation. This choice may also result in the coordinate y value being smaller than the x coordinate value, but this is also largely immaterial barring convention. With this in mind, (A1.8) will be solved for Q, β and y using relations (A1.9), and so the coordinate y is instantly fixed by k and l in (A1.9d).

The general solution to the Diophantine equation (A1.9a) has already been evaluated in (A1.4). However, does the arbitrary integer m in (A1.4) have to have the same value? In fact, for consistency in transformations (beyond the scope of this paper), it actually has to be $-m$; the particular solution remains the same two integers s' and t' (A1.4).

Substituting for s and t from (A1.4) into (A1.9c) derives β

(A1.11) $\beta = t'^2 - s'^2 - 2m(ls' - kt') - m^2(l^2 - k^2)$.

To derive Q, substitute for s and t (A1.4) into (A1.9b).

(A1.12a) $Q = (ls' - kt') + m(l^2 - k^2)$.

Denoting a particular solution for Q by Q', where

(A1.12b) $Q' = (ls' - kt')$,

and using (A1.9d) to substitute for $(l^2 - k^2)$ in terms of y, the general solution for Q (A1.12a) can be re-written

(A1.12c) $Q = Q' + my$.

This equation shows that Q is arbitrary to within an integer multiple integer m of y.

Denoting a particular solution for β as β'

(A1.13a) $\beta' = t'^2 - s'^2$,

and using (A1.9d) for y, and (A1.12b) for Q', the general solution for β (A1.11) is re-expressed as

(A1.13b) $\beta = \beta' - 2mQ' - m^2 y$.

Lastly, variables P, α and x have to be obtained.

With y given by (A1.9d) and z by (A1.3d), x can be obtained directly using the Pythagoras equation (2.10)

(A1.14a) $x^2 = z^2 - y^2 = (l^2 + k^2)^2 - (l^2 - k^2)^2$.

Taking the positive root for x, this simplifies to

(A1.14b) $x = 2kl$.

It is noted that the parameter solutions for x (A1.14b), y (A1.9d) and z (A1.3d), are the familiar parameterisations often used when expressing the solutions to Pythagoras (2.10).

With x now determined, justification for the sign allocation of a minus to R in (A1.3b) can be given using equation $x = Ry + Qz$ (2.9a) for x. Substituting for Q (A1.9b), R (A1.3b), y (A1.9d) and z (A1.3d), then (2.9a) becomes

(A1.14c) $$x = -(ls + kt)(l^2 - k^2) + (ls - kt)(l^2 + k^2),$$

Expanding the brackets, and using (A1.9a) for $(ks - lt)$, this simplifies to $x = 2kl$, which is as expected according to (A1.14b). If R were not given the sign choice in (A1.3b), then the expression for x actually evaluates to $x = 2sl^3 - 2tk^3$ which contradicts (A1.14b).

Having obtained x, the equation $0 = yQ + Rz - xP$ (9.5) is used to obtain P. Substituting for x (A1.14b), y (A1.9d), z (A1.3d), Q (A1.12a) and R (A1.6a) into (9.2), and rearranging to give P

(A1.15a)
$$2klP = (l^2 - k^2)(ls' - kt') - (l^2 + k^2)(ls' + kt') + m(l^2 - k^2)^2 - m(l^2 + k^2)^2,$$

which tidies to

(A1.15b) $$P = -(ks' + lt') - m2kl.$$

Substituting for s' and t' from (A1.4), this is also written in general terms as

(A1.15c) $$P = -(ks + lt).$$

Denoting a particular solution for P by P' where

(A1.15d) $$-P' = (ks' + lt'),$$

and using (A1.14b) to substitute for $2lk$ in terms of x, the most general solution becomes

(A1.15e) $$P = P' - mx.$$

This equation shows that P is arbitrary to within integer multiples m of x.

To obtain α, an expression for P^2 is first obtained by substitution for Q (A1.9b) and R (A1.3b) into the dynamical conservation equation, $+1 = P^2 + Q^2 - R^2$ (2.5), to give

(A1.15f) $$P^2 = 1 + 4klst.$$

This makes for a simple evaluation of α by substitution for P^2 from (A1.15f) into $\alpha x = (1 - P^2)$ (3.1a), to give, using $x = 2kl$ (A1.14b), (A1.16a) $\alpha = -2st$.

Substituting for s and t from (A1.4), the general solution for α is

(A1.16b) $\alpha = -2s't' - 2m(ks' + lt') - 2m^2 lk$.

Denoting a particular solution for α as α' then

(A1.16c) $\alpha' = -2s't'$,

and using (A1.15d) for P', and (A1.14b) for x, the general solution for α (A1.16b) can be re-written as

(A1.16d) $\alpha = \alpha' + 2mP' - m^2 x$.

This completes the analytic solution for all nine variables $\{x, y, z, P, Q, R, \alpha, \beta, \gamma\}$ and thus solutions to the dynamical conservation equation (2.5) and the two Pythagoras equations (2.10) and (2.14). Example data is provided in Appendix (E).

Lastly, it is noted this analytic solution does not give all Pythagorean triples and Appendix (D) is devoted to extending the arguments to cover every possible Pythagorean triple in accordance with definition (1.0).

Appendix (B)
Key Equation summary

Matrix $\mathbf{A}$ in 'dynamical', integer variables P, Q, R

(1.1) $$\mathbf{A} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix},\ P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0).$$

Dynamical conservation equation

(2.5) $$+1 = P^2 + Q^2 - R^2.$$

Pythagoras equation for (x, y, z) and (α, β, γ)

(2.10) $$0 = x^2 + y^2 - z^2$$
(2.14) $$0 = \alpha^2 + \beta^2 - \gamma^2.$$

Divisibility factors defining equations

(3.1a) $$\alpha x = (1 - P^2)$$
(3.1b) $$\beta y = (1 - Q^2)$$
(3.1c) $$\gamma z = (1 + R^2).$$

Analytic solution for all $\{x, y, z, P, Q, R, \alpha, \beta, \gamma\}$

Arbitrary integers k and l subject to the conditions

(A1.2) $$k, l \in \mathbb{Z},\ (k, l) \neq (0,0),\ \gcd(k, l) = 1.$$

Parametric solution for x, y, z

(A1.14b) $$x = 2kl$$
(A1.9d) $$y = (l^2 - k^2)$$
(A1.3d) $$z = (l^2 + k^2).$$

Linear Diophantine Equation in integers s and t

(A1.3a) $$+1 = ks - lt,\ s, t \in \mathbb{Z}.$$

Particular, integer solution s' and t', and general solutions s and t parameterised by arbitrary, integer m

(A1.4a) $s = s' + ml$
(A1.4b) $t = t' + mk$.

Dynamical variables (P, Q, R)

(A1.15c) $P = -(ks + lt)$,
(A1.9b) $Q = (ls - kt)$
(A1.3b) $R = -(ls + kt)$.

Divisibility factors (α, β, γ)

(A1.16a) $\alpha = -2st$
(A1.9c) $\beta = (t^2 - s^2)$
(A1.3c) $\gamma = (t^2 + s^2)$.

Eigenvalues

(2.7) $\lambda = +1$, $\lambda = 0$, $\lambda = -1$.

Standard Eigenvectors

(2.11) $\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, (2.15) $\mathbf{X}_0 = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix}$, (2.12) $\mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}$.

Conjugate eigenvectors

(8.1) $\mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$
(8.2) $\mathbf{X}^0 = \begin{pmatrix} P & -Q & -R \end{pmatrix}$
(8.3) $\mathbf{X}^- = \begin{pmatrix} x & y & -z \end{pmatrix}$.

Eigenvector Equations

(2.8a) $\mathbf{AX}_+ = \mathbf{X}_+$, $\lambda = +1$
(2.8b) $\mathbf{AX}_0 = 0$, $\lambda = 0$
(2.8c) $\mathbf{AX}_- = -\mathbf{X}_-$, $\lambda = -1$.

Residual matrices

(5.3) $\mathbf{E}_+ = (\alpha\mathbf{X}_+, \beta\mathbf{X}_+, \gamma\mathbf{X}_+)$

(6.2) $\mathbf{E}_0 = (-P\mathbf{X}_0, Q\mathbf{X}_0, R\mathbf{X}_0)$

(7.3) $\mathbf{E}_- = (x\mathbf{X}_-, y\mathbf{X}_-, -z\mathbf{X}_-)$.

Residual matrices as dyadic products

(9.14) $\mathbf{E}_+ = \mathbf{X}_+\mathbf{X}^+$

(9.15) $\mathbf{E}_0 = -\mathbf{X}_0\mathbf{X}^0$

(9.16) $\mathbf{E}_- = \mathbf{X}_-\mathbf{X}^-$.

Eigenvector dot product, invariants

(9.1) $\mathbf{X}_+ \cdot \mathbf{X}^- = x^2 + y^2 - z^2 = 0$

(9.2) $\mathbf{X}_- \cdot \mathbf{X}^+ = \alpha^2 + \beta^2 - \gamma^2 = 0$

(9.3) $\mathbf{X}_0 \cdot \mathbf{X}^0 = P^2 + Q^2 - R^2 = +1$

(9.4) $\mathbf{X}_+ \cdot \mathbf{X}^+ = \mathbf{X}_- \cdot \mathbf{X}^- = \alpha x + \beta y + \gamma z = +2$

(9.5) $\mathbf{X}_+ \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^- = xP - yQ - zR = 0$

(9.6) $\mathbf{X}_- \cdot \mathbf{X}^0 = \mathbf{X}_0 \cdot \mathbf{X}^+ = \alpha P - \beta Q + \gamma R = 0$

Eigenvector cross products

(9.8) $\mathbf{X}_+ \wedge \mathbf{X}_+ = \mathbf{X}_0 \wedge \mathbf{X}_0 = \mathbf{X}_- \wedge \mathbf{X}_- = 0$

$\mathbf{X}^+ \wedge \mathbf{X}^+ = \mathbf{X}^0 \wedge \mathbf{X}^0 = \mathbf{X}^- \wedge \mathbf{X}^- = 0$

(9.9) $\mathbf{X}^+ \wedge \mathbf{X}^0 = -\mathbf{X}^0 \wedge \mathbf{X}^+ = \mathbf{X}_-$

$\mathbf{X}_+ \wedge \mathbf{X}_0 = -\mathbf{X}_0 \wedge \mathbf{X}_+ = \mathbf{X}^-$

(9.10) $\mathbf{X}^- \wedge \mathbf{X}^+ = -\mathbf{X}^+ \wedge \mathbf{X}^- = +2\mathbf{X}_0$

$\mathbf{X}_- \wedge \mathbf{X}_+ = -\mathbf{X}_+ \wedge \mathbf{X}_- = +2\mathbf{X}^0$

(9.11) $\mathbf{X}^0 \wedge \mathbf{X}^- = -\mathbf{X}^- \wedge \mathbf{X}^0 = \mathbf{X}_+$

$\mathbf{X}_0 \wedge \mathbf{X}_- = -\mathbf{X}_- \wedge \mathbf{X}_0 = \mathbf{X}^+$.

The eigenvector triple product is also an invariant with a value of $+2$

(9.12) $\mathbf{X}^+ \wedge \mathbf{X}^0 \cdot \mathbf{X}^- = \mathbf{X}_- \cdot \mathbf{X}^- = \mathbf{X}_+ \wedge \mathbf{X}_0 \cdot \mathbf{X}_- = \mathbf{X}^- \cdot \mathbf{X}_- = +2$

Appendix (C)
Example. Pythagorean Triple (4,3,5)

Choose integers k and l subject to (A1.2)

(C1.1) $l = 2$, $k = 1$.

The triple (x, y, z) is then given by the familiar Pythagoras parameterisations $x = 2kl$ (A1.14b), $y = (l^2 - k^2)$ (A1.9d), and $z = (l^2 + k^2)$ (A1.3d)

(C1.2) $x = 4$, $y = 3$, $z = 5$.

Solve the linear Diophantine equation $+1 = ks - lt$ (A1.3a), to give a general solution for s, t in terms of an arbitrary integer parameter m

(C1.3) $s = 1 + 2m$, $t = m$.

The triple (P, Q, R) can then be obtained from $P = -(ks + lt)$ (A1.15c), $Q = (ls - kt)$ (A1.9b), and $R = -(ls + kt)$ (A1.3b), also parameterised by m

(C1.4) $P = -1 - 4m$
(C1.5) $Q = +2 + 3m$
(C1.6) $R = -2 - 5m$.

The divisibility factor triple (α, β, γ) is obtained from $\alpha = -2st$ (A1.16a), $\beta = (t^2 - s^2)$ (A1.9c), and $\gamma = (t^2 + s^2)$ (A1.3c), parameterised by m

(C1.7) $\alpha = -(4m^2 + 2m)$
(C1.8) $\beta = -(3m^2 + 4m + 1)$
(C1.9) $\gamma = +(5m^2 + 4m + 1)$.

For the primitive solution $m = 0$ and so s and t in (C1.3) become

(C1.10) $s = +1$, $t = 0$.

Substituting $m = 0$ into (C1.4) to (C1.9), the following values for the dynamical variables and scale factors are obtained

(C1.11a) $P = -1$, $Q = +2$, $R = -2$
(C1.11b) $\alpha = 0$, $\beta = -1$, $\gamma = 1$.

It is seen that the triple (α, β, γ) is almost trivial in so far as $\alpha = 0$ and hence $|\beta| = |\gamma|$. As discussed earlier in section (10), this situation can be remedied, if so desired, by varying free parameter m - using even values only to obtain primitive triples. For example, substituting $m = 2$ into (C1.4) to (C1.9), gives the primitive triple $(\alpha,\beta,\gamma) = (-20,-21,+29)$ with dynamical variables $(P,Q,R) = (-9,+8,-12)$. However, continuing with $(\alpha,\beta,\gamma) = (0,-1,+1)$, then with all variables assigned, (C1.2), (C1.11a) and (C1.11b), the eigenvectors are

(C1.12) $$\mathbf{X}_{+} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix},\ \mathbf{X}_{0} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix},\ \mathbf{X}_{-} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix},$$

(C1.13) $$\mathbf{X}^{+} = \begin{pmatrix} 0 & -1 & +1 \end{pmatrix},\ \mathbf{X}^{0} = \begin{pmatrix} -1 & -2 & +2 \end{pmatrix},\ \mathbf{X}^{-} = \begin{pmatrix} 4 & 3 & -5 \end{pmatrix}.$$

Using the values for P,Q,R (C1.11a), the matrices $\mathbf{A}$ (1.1) and $\mathbf{A}^2$ are

(C1.14) $$\mathbf{A} = \begin{pmatrix} 0 & -2 & +2 \\ +2 & 0 & -1 \\ +2 & -1 & 0 \end{pmatrix}, \qquad \mathbf{A}^2 = \begin{pmatrix} 0 & -2 & 2 \\ -2 & -3 & 4 \\ -2 & -4 & 5 \end{pmatrix}.$$

Using $\mathbf{A}$ and $\mathbf{A}^2$, the residual matrices $\mathbf{E}_{+}$, $\mathbf{E}_{0}$, $\mathbf{E}_{-}$ are calculated from (4.2) as

(C1.15a) $$\mathbf{E}_{+} = \begin{pmatrix} 0 & -4 & +4 \\ 0 & -3 & +3 \\ 0 & -5 & +5 \end{pmatrix},$$

(C1.15b) $$\mathbf{E}_{+} = (0\mathbf{X}_{+}, -1.\mathbf{X}_{+}, +1\mathbf{X}_{+}),$$

(C1.16a) $$\mathbf{E}_{-} = \begin{pmatrix} 0 & 0 & 0 \\ -4 & -3 & +5 \\ -4 & -3 & +5 \end{pmatrix},$$

(C1.16b) $$\mathbf{E}_{-} = (4\mathbf{X}_{-}, 3\mathbf{X}_{-}, -5\mathbf{X}_{-}),$$

(C1.17a) $$\mathbf{E}_{0} = \begin{pmatrix} -1 & -2 & +2 \\ -2 & -4 & +4 \\ -2 & -4 & +4 \end{pmatrix},$$

(C1.17b) $$\mathbf{E}_{0} = (+1\mathbf{X}_{0}, +2\mathbf{X}_{0}, -2\mathbf{X}_{0}).$$

Appendix (D)
An extension to all Pythagorean triples

The analytic solution for (x, y, z), as given by $x = 2kl$ (A1.14b), $y = (l^2 - k^2)$ (A1.9d) and $z = (l^2 + k^2)$ (A1.3d), and subject to the condition $(k,l) \neq (0,0)$ and $\gcd(k,l) = 1$, (A1.2), is known not to give all Pythagorean triples according to definition (1.0). Specifically, there are three missing cases:

Case 1) if it generates the triple (x, y, z) then it will not generate the triple (y, x, z);

Case 2) it does not span the complete set of non-primitive triples, i.e. it does not give all triples (Cx, Cy, Cz) for arbitrary, non-zero, integer constant C, where (x, y, z) is a primitive Pythagorean triple;

Case 3) it cannot give a negative value for z, regardless of the values of k and l.

(D1) Case 1. This first case has already been mentioned in section (10) and is trivially resolved by swapping the solution for y (A1.9d) with x (A1.14b) when making the decision whether to solve for dynamical variable P or Q first, after obtaining R, see Appendix (A), equation (A1.8) onward. It is also necessary to swap the solutions for Q with P and β with α at the same time such that the association of x with (P, α), and y with (Q, β), is preserved and all equations can be used, as defined, without any need to swap terms.

(D2) Case 2. Because the analytic solution gives all primitive triples (x, y, z), and since each triple forms an eigenvector $\mathbf{X}_+$ (2.11) for eigenvalue $\lambda = +1$, eigenvector equation (2.8a), then $\mathbf{X}_+$, and consequently (x, y, z), is arbitrary to within a scale factor. So, for non-zero, integer constant C, the vector $C\mathbf{X}_+$ is also a valid solution to (2.8a) for eigenvalue $\lambda = +1$. Consequently, every non-primitive, Pythagorean triple (Cx, Cy, Cz) can also be associated with the matrix $\mathbf{A}$, eigenvalue $\lambda = +1$.

The same argument also applies to the other Pythagorean triple (α, β, γ), eigenvector $\mathbf{X}_-$ (2.12), eigenvalue $\lambda = -1$, eigenvector equation (2.8c). However, as already mentioned in section (10), the triple (α, β, γ) itself may already be non-primitive.

As far as this paper is concerned, the argument above for $\mathbf{X}_+$ is sufficient for the conclusion reached in section (12). However, it should be noted that the scaling of eigenvectors, e.g. $C\mathbf{X}_+$ and $C\mathbf{X}_-$, does affect the non-zero invariants, e.g. (9.3) and (9.4). The subject is generally beyond the scope of this paper but, for advance information only, a consistent scaling can actually be applied to all three eigenvectors $C\mathbf{X}_+$, $C\mathbf{X}_0$, $C\mathbf{X}_-$ and their conjugates $C\mathbf{X}^+$, $C\mathbf{X}^0$ and $C\mathbf{X}^-$, with a consequential re-scaling of the invariants. This does mean that invariant quantities only remain invariant for all eigenvectors in the solution space for a fixed value of C, and change if the value of C is changed. Nevertheless, given the solution space is infinite for each value of C, the invariant quantities are not without significance.

For an example of eigenvector scaling, the two non-zero invariants, given by (9.3) and (9.4), become

(D2.1) $$C\mathbf{X}_0 \cdot C\mathbf{X}^0 = C^2P^2 + C^2Q^2 - C^2R^2 = +C^2$$

(D2.2) $$C\mathbf{X}_+ \cdot C\mathbf{X}^+ = (C\alpha)(Cx) + (C\beta)(Cy) + (C\gamma)(Cz) = +2C^2 .$$

The vector triple products scale by C^3 since they involve a product of three vectors. For example, (9.12) becomes

(D2.4) $$C\mathbf{X}^+ \wedge C\mathbf{X}^0 \cdot C\mathbf{X}^- = +2C^3 .$$

The topic of scaling is more advanced, and rescaling invariants is not the only issue. A second issue arises when examining equations (2.9), (2.16) and (3.1). These no longer hold true if a direct substitution of the linearly scaled variables is made for the original, un-scaled variables. For example, divisibility factor α is defined by (3.1a) as $\alpha x = (1 - P^2)$, and replacing all variables by their scaled equivalents gives

(D2.5) $$(C\alpha)(Cx) = (1 - (CP)^2) .$$

Since the primitive (un-scaled) variables α, x and P satisfy the defining equation for α, i.e. $\alpha x = (1 - P^2)$ (3.1a), then (D2.5), in its scaled form above, cannot also be satisfied in the general case when $|C| > 1$. In fact, this is remedied by replacing the +1 term in (D2.5) by C^2 to give the following

(D2.6) $$(C\alpha)(Cx) = (C^2 - (CP)^2) .$$

It is seen that a factor of C^2 can now be divided throughout (D2.6) to give and the original, un-scaled definition.

This problem, and the need to modify invariants, is directly connected with non-unity eigenvalues, $\lambda = \pm C$, $|C| > 1$. But, as stated, this topic is beyond the scope of this paper and the information provided here is merely to clarify any seemingly unresolved issues.

(D3) Case 3. Whilst a negative value for k or l (not both) can be used to obtain a negative value of x, and making $|k| > |l|$ can be used to make a negative value of y by (A1.9d), it is not possible to get a negative value of z by changing either k or l since $z = (l^2 + k^2)$ (A1.3d). However, the aforementioned case 2 rescues the situation since the scaling constant C can be made negative, e.g. $C = -1$, to simultaneously reverse the sign of all elements (x, y, z) of $\mathbf{X}_+$. Note that since $|C| = 1$, this will not alter any invariants. If it is also desired, the sign and/or relative magnitude of arbitrary integers k or l can then be separately modified to achieve any desired sign combination of the elements x and y, whilst keeping z negative. It is therefore concluded every sign combination of the elements of the eigenvector $\mathbf{X}_+$ can be obtained by suitable choice of k, l and C.

Appendix (E)
Example Data

The following table gives the complete, analytic solution for all variables, for values of k, l (A1.2) from $l = 2$ to $l = 20$ and $k = 1$ to $k = l$, for each value of k, where $k < l$, $\gcd(k, l) = 1$ and, additionally, $l - k \equiv 1 (\mathrm{mod}\, 2)$, i.e. only an odd k, even l combination or vice versa.

The list is sorted by l value, then k value, which does not give the conventional listing of smallest value of x, in a Pythagorean triple (x, y, z), first. The x value is always even, as per the solution (A1.14b) and the y value (A1.9d) always odd. The list covers all the Pythagorean triples $x = 1..40$, plus others. But due to the way it is computer generated on k and l, the triple $(41,840,841)$ is the first not to appear, albeit there are higher triples present such as $(44,117,125)$.

All variables are calculated according to the analytic solution in Appendix (A) with the appropriate equations reproduced below.

(A1.14b) $x = 2kl$
(A1.9d) $y = (l^2 - k^2)$
(A1.3d) $z = (l^2 + k^2)$

(A1.3a) $+1 = ks - lt$

(A1.15c) $P = -(ks + lt)$
(A1.9b) $Q = (ls - kt)$
(A1.3b) $R = -(ls + kt)$

(A1.16a) $\alpha = -2st$
(A1.9c) $\beta = (t^2 - s^2)$
(A1.3c) $\gamma = (t^2 + s^2)$.

l	k	x	y	z	s	t	P	$-Q$	R	α	β	$-\gamma$
2	1	4	3	5	1	0	-1	-2	-2	0	-1	-1
3	2	12	5	13	2	1	-7	-4	-8	-4	-3	-5
4	1	8	15	17	1	0	-1	-4	-4	0	-1	-1
4	3	24	7	25	3	2	-17	-6	-18	-12	-5	-13
5	2	20	21	29	3	1	-11	-13	-17	-6	-8	-10
5	4	40	9	41	4	3	-31	-8	-32	-24	-7	-25
6	1	12	35	37	1	0	-1	-6	-6	0	-1	-1
6	5	60	11	61	5	4	-49	-10	-50	-40	-9	-41
7	2	28	45	53	4	1	-15	-26	-30	-8	-15	-17
7	4	56	33	65	2	1	-15	-10	-18	-4	-3	-5
7	6	84	13	85	6	5	-71	-12	-72	-60	-11	-61
8	1	16	63	65	1	0	-1	-8	-8	0	-1	-1
8	3	48	55	73	3	1	-17	-21	-27	-6	-8	-10
8	5	80	39	89	5	3	-49	-25	-55	-30	-16	-34

8	7	112	15	113	7	6	-97	-14	-98	-84	-13	-85
9	2	36	77	85	5	1	-19	-43	-47	-10	-24	-26
9	4	72	65	97	7	3	-55	-51	-75	-42	-40	-58
9	8	144	17	145	8	7	-127	-16	-128	-112	-15	-113
10	1	20	99	101	1	0	-1	-10	-10	0	-1	-1
10	3	60	91	109	7	2	-41	-64	-76	-28	-45	-53
10	7	140	51	149	3	2	-41	-16	-44	-12	-5	-13
10	9	180	19	181	9	8	-161	-18	-162	-144	-17	-145
11	2	44	117	125	6	1	-23	-64	-68	-12	-35	-37
11	4	88	105	137	3	1	-23	-29	-37	-6	-8	-10
11	6	132	85	157	2	1	-23	-16	-28	-4	-3	-5
11	8	176	57	185	7	5	-111	-37	-117	-70	-24	-74
11	10	220	21	221	10	9	-199	-20	-200	-180	-19	-181
12	1	24	143	145	1	0	-1	-12	-12	0	-1	-1
12	5	120	119	169	5	2	-49	-50	-70	-20	-21	-29
12	7	168	95	193	7	4	-97	-56	-112	-56	-33	-65
12	11	264	23	265	11	10	-241	-22	-242	-220	-21	-221
13	2	52	165	173	7	1	-27	-89	-93	-14	-48	-50
13	4	104	153	185	10	3	-79	-118	-142	-60	-91	-109
13	6	156	133	205	11	5	-131	-113	-173	-110	-96	-146
13	8	208	105	233	5	3	-79	-41	-89	-30	-16	-34
13	10	260	69	269	4	3	-79	-22	-82	-24	-7	-25
13	12	312	25	313	12	11	-287	-24	-288	-264	-23	-265
14	1	28	195	197	1	0	-1	-14	-14	0	-1	-1
14	3	84	187	205	5	1	-29	-67	-73	-10	-24	-26
14	5	140	171	221	3	1	-29	-37	-47	-6	-8	-10
14	9	252	115	277	11	7	-197	-91	-217	-154	-72	-170
14	11	308	75	317	9	7	-197	-49	-203	-126	-32	-130
14	13	364	27	365	13	12	-337	-26	-338	-312	-25	-313
15	2	60	221	229	8	1	-31	-118	-122	-16	-63	-65
15	4	120	209	241	4	1	-31	-56	-64	-8	-15	-17
15	8	240	161	289	2	1	-31	-22	-38	-4	-3	-5
15	14	420	29	421	14	13	-391	-28	-392	-364	-27	-365
16	1	32	255	257	1	0	-1	-16	-16	0	-1	-1
16	3	96	247	265	11	2	-65	-170	-182	-44	-117	-125
16	5	160	231	281	13	4	-129	-188	-228	-104	-153	-185
16	7	224	207	305	7	3	-97	-91	-133	-42	-40	-58
16	9	288	175	337	9	5	-161	-99	-189	-90	-56	-106
16	11	352	135	377	3	2	-65	-26	-70	-12	-5	-13
16	13	416	87	425	5	4	-129	-28	-132	-40	-9	-41
16	15	480	31	481	15	14	-449	-30	-450	-420	-29	-421
17	2	68	285	293	9	1	-35	-151	-155	-18	-80	-82
17	4	136	273	305	13	3	-103	-209	-233	-78	-160	-178
17	6	204	253	325	3	1	-35	-45	-57	-6	-8	-10
17	8	272	225	353	15	7	-239	-199	-311	-210	-176	-274
17	10	340	189	389	12	7	-239	-134	-274	-168	-95	-193
17	12	408	145	433	10	7	-239	-86	-254	-140	-51	-149
17	14	476	93	485	11	9	-307	-61	-313	-198	-40	-202
17	16	544	33	545	16	15	-511	-32	-512	-480	-31	-481
18	1	36	323	325	1	0	-1	-18	-18	0	-1	-1
18	5	180	299	349	11	3	-109	-183	-213	-66	-112	-130
18	7	252	275	373	13	5	-181	-199	-269	-130	-144	-194
18	11	396	203	445	5	3	-109	-57	-123	-30	-16	-34
18	13	468	155	493	7	5	-181	-61	-191	-70	-24	-74
18	17	612	35	613	17	16	-577	-34	-578	-544	-33	-545
19	2	76	357	365	10	1	-39	-188	-192	-20	-99	-101
19	4	152	345	377	5	1	-39	-91	-99	-10	-24	-26
19	6	228	325	397	16	5	-191	-274	-334	-160	-231	-281
19	8	304	297	425	12	5	-191	-188	-268	-120	-119	-169
19	10	380	261	461	2	1	-39	-28	-48	-4	-3	-5
19	12	456	217	505	8	5	-191	-92	-212	-80	-39	-89

19	14	532	165	557	15	11	-419	-131	-439	-330	-104	-346
19	16	608	105	617	6	5	-191	-34	-194	-60	-11	-61
19	18	684	37	685	18	17	-647	-36	-648	-612	-35	-613
20	1	40	399	401	1	0	-1	-20	-20	0	-1	-1
20	3	120	391	409	7	1	-41	-137	-143	-14	-48	-50
20	7	280	351	449	3	1	-41	-53	-67	-6	-8	-10
20	9	360	319	481	9	4	-161	-144	-216	-72	-65	-97
20	11	440	279	521	11	6	-241	-154	-286	-132	-85	-157
20	13	520	231	569	17	11	-441	-197	-483	-374	-168	-410
20	17	680	111	689	13	11	-441	-73	-447	-286	-48	-290
20	19	760	39	761	19	18	-721	-38	-722	-684	-37	-685