Quaternions and Angular Dynamics Notes

An edited extract from

Unity Root Matrix Theory
Mathematical and Physical Advances Volume II

Sections 9 to 12 and Appendix H

R J Miller
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References

[1] Unity Root Matrix Theory, Physics in Integers, R. J. Miller, FastPrint Publishing, 2011, ISBN 978-184426-974-7,

http://www.fast-print.net/bookshop/823/unity-root-matrix-theory-physics-in-integers

This book is broken into six separate papers, each paper is given a specific reference #1 to #6 as follows:

[1]#1 Unity Root Matrix Theory Foundations
[1]#2 see [11], below
[1]#3 Geometric and Physical Aspects
[1]#4 Solving Unity Root Matrix Theory
[1]#5 Unifying Concepts
[1]#6 A Non-unity Eigenvalue

[2] Unity Root Matrix Theory, Higher Dimensional Extensions, R. J. Miller, FastPrint Publishing, 2012, ISBN 978-178035-296-1,

http://www.fast-print.net/bookshop/1007/unity-root-matrix-theory-higher-dimensional-extensions

[3] Unity Root Matrix Theory, Mathematical and Physical Advances Volume I, Bright Pen, 2013, ISBN 978-0-7552-1535-5,

http://www.authorsonline.co.uk/bookshop.php?act=bookview&id=1338

[4] not applicable

[5] Sadri Hassani, Foundations of Mathematical Physics, Prentice-Hall International Editions, 1991, ISBN 0-13-327503-5.

[6] not applicable

[7] Unity Root Matrix Theory web site, http://www.urmt.org.

[8] Roger Penrose, The Road to Reality, Jonathan Cape London 2004. ISBN 0-224-04447-8. This is selected as a general, readable reference that, fortunately, does not shy away from equations. It covers many different topics mentioned throughout this book.

[9] not applicable

[10] Compactification of an n-dimensional eigenvector space over long evolutionary timescales. R. J. Miller, 2011, a free PDF document available for download. This paper is equivalent to Section (14) in [2], i.e. Ref. [2],14.

http://www.urmt.org/urmt_dimensional_compactification.pdf

[11] Pythagorean Triples as Eigenvectors and Related Invariants, R. J. Miller, 2010, a free PDF document available for download:

http://www.urmt.org/pythag_eigenvectors_invariants.pdf

This paper is equivalent to Paper 2 in [1], i.e. Ref. [1]#2.

[12] not applicable.

[13] H Goldstein, Classical Mechanics, Adison-Wesley, 1980, 978-0-20102-918-5. There is a later, revised edition: Goldstein Safko Poole, Classical Mechanics, Pearson Education Ltd. 978-1-29202-655-8, but the original is the classic work.

[14] Venzo de Sabatta and Bidyut Kumar Datta, Geometric Algebra and Applications to Physics, Taylor & Francis, 978-1-58488-722-9; see also [17].

[15] Jack B. Kuipers, Quaternions and Rotation Sequences, A Primer with Applications to Orbits, Aerospace and Virtual Reality, Princeton University Press, ISBN 978-0-691-10298-6.

[16] not applicable

[17] D. Hestenes, Space-Time Algebra, Gordon and Breach, New York, 1966. This appears to be the founding work on the use of Geometric algebra in Physics; search 'D Hestenes' on the web for much more, later, related material.

[18] Unity Root Matrix Theory and the Riemann Hypothesis, R. J. Miller, 2013, a free PDF document available for download:

http://www.urmt.org/urmt_riemann_link.pdf

[19] not applicable

Example reference syntax used throughout:

[1]#3,8 : Book I, Paper #3, Section (8)
[2],7 : Book 2, Section (7)
[2],7.2 : Book 2, Section (7), equation (7.2)
[2],1-6 : Book 2, sub-section (1-6)
[2],E8 : Book 2, Appendix E, equation (E8)
(7-4) : this book, sub-section (7-4)
(7.41) : this book, Section (7), equation (7.41)
(A) : this book Appendix (A)
(A1a) : this book Appendix (A), equation (A1a)

As regards existing, published work in the field of Unity Root Matrix Theory, since this is a relatively new subject area (less than ten years old) then, to the author's best knowledge, the only currently available texts are the first three published books, [1]-[3], plus some free material, e.g. [10], [11], [18]. The reader may wish to visit the web-site [7] occasionally since new, free material is added every few months.

Acronyms and Abbreviations

AVE : Arbitrary Vector Embedding
DCE : Dynamical Conservation Equation
URMn the $n \times n$ matrix formulation of URMT
URMT : Unity Root Matrix Theory

Table of Contents

9 Algebraic Fundamentals of Quaternions 1
10 The Quaternion Representation Matrix 13
11 Quaternions and Single Axis Rotations 25
12 Quaternions and Three Axis Rotations 33
24 Appendix (H) Rotations and Angular Dynamics 43

Part IV

Quaternions

9 Algebraic Fundamentals of Quaternions

(9-1) Introduction

This section is a basic introduction to quaternions and their algebraic properties. Together with the next section, it serves as a prelude to the treatment of rotations given in Sections (11) and (12).

Quaternions were introduced around 1843 by W J Hamilton in an attempt to describe rotations in three dimensions by analogy with the usage of complex numbers in describing planar rotations about a single axis. Quaternions are described in a wealth of literature, and [8] offers a general descriptive overview under their generic name of 'hypercomplex' numbers. This text is primarily mentioned because it also relates to Clifford and Grassman algebras, in particular, exterior (or wedge/Grassman) products similar to those used in URMT's exterior product formulation given earlier in Sections (4) to (6); see [15] for a more detailed mathematical account of quaternions. Of course, there is also plenty of free material on the web. A relatively brief account is given here, which should be sufficient for the URMT material that follows.

(9-2) Formal Definition of a Quaternion

By defining three numbers (base elements) i, j, k with the following properties:

(9.1)

(9.1a) $i^2 = j^2 = k^2 = -1$

(9.1b) $ij = -ji = k$

(9.1c) $jk = -kj = i$

(9.1d) $ki = -ik = j$,

and four arbitrary real numbers q_0, q_1, q_2, q_3,

$$q_0, q_1, q_2, q_3 \in \mathbb{R},$$

then a quaternion $\mathbf{q}$ is defined in terms i, j, k and q_0, q_1, q_2, q_3 as the following number:

(9.2) $\mathbf{q} = q_0 + q_1 i + q_2 j + q_3 k$.

Each of the three numbers i, j, k are similar to the complex number i because they square to -1 by definition (9.1a). However, there are now three of them and they do not commute, e.g. $ij \neq ji$, albeit they do commute with real numbers, i.e.

(9.3) $xi = ix$, $yj = jy$, $zk = kz$, $x, y, z \in \mathbb{R}$,

Thus, by (9.3) and definition (9.2), quaternions commute with real numbers, i.e.

(9.4) $k\mathbf{q} = \mathbf{q}k$, $k \in \mathbb{R}$.

Because of the non-commutativity amongst i, j, k, the quaternions themselves do not commute, unlike complex numbers, discussed shortly below.

The i, j, k are also termed 'bases', and can be thought of as similar to the unit basis vectors 'i, j, k' in standard Cartesian vector algebra, where a vector $\underline{\mathbf{r}}$, with three real components, is defined in terms of the basis vectors as

(9.5) $\underline{\mathbf{r}} = xi + yj + zk$, $x, y, z \in \mathbb{R}$.

Indeed, the i, j, k component $q_1 i + q_2 j + q_3 k$ of the quaternion $\mathbf{q}$ (9.2) is termed the vector part. However, this is really where the similarity ends. For instance, in vector algebra the i, j, k never multiply out according to the quaternion multiplication rules (9.1). Instead, in standard vector analysis, a product such as ij is an outer product - a product of two vectors that gives a rank-two result, and this product is most definitely not equal to the basis vector k as in (9.1b) - a vector being an object of rank one.

Nevertheless, it is still very useful to split the quaternion $\mathbf{q}$ into a real, scalar part q_0 and a 'vector' part $\underline{\mathbf{q}}$ defined as

(9.6) $\underline{\mathbf{q}} = q_1 i + q_2 j + q_3 k$,

so that the complete quaternion is now written as

(9.7) $\mathbf{q} = q_0 + \underline{\mathbf{q}}$.

The vector part is also written in the standard, column vector form as

(9.8) $\underline{\mathbf{q}} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$,

and its transpose is simply the row vector

(9.9) $\underline{\mathbf{q}}^T = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix}$.

The entire, four-element quaternion is then written as

(9.10) $\mathbf{q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$ or $\mathbf{q} = \begin{pmatrix} q_0 \\ \underline{\mathbf{q}} \end{pmatrix}$, $\mathbf{q}^T = \begin{pmatrix} q_0 & \underline{\mathbf{q}}^T \end{pmatrix}$.

A quaternion may comprise just a vector part $\underline{\mathbf{q}}$ with the scalar term q_0 zero, i.e.

(9.11) $\mathbf{q} = \begin{pmatrix} 0 \\ \underline{\mathbf{q}} \end{pmatrix}, \; q_0 = 0$,

and, likewise, a quaternion can comprise a real-only part, with the vector part simply the null, zero vector, i.e.

(9.12) $\mathbf{q} = \begin{pmatrix} q_0 \\ 0 \end{pmatrix}, \; \underline{\mathbf{q}} = 0$.

This latter quaternion is just a real number q_0, and can be treated as such in all algebraic manipulation, just like a complex number with no imaginary (complex) component.

In between these two extremes lies the possibility for a quaternion to represent a complex number by zeroing any two of the three vector components q_1, q_2, q_3, i.e.

(9.13)

$$\mathbf{q} = q_0 + q_1 i \,, \; q_2, q_3 = 0$$

$$\mathbf{q} = q_0 + q_2 j \,, \; q_1, q_3 = 0$$

$$\mathbf{q} = q_0 + q_3 k \,, \; q_1, q_2 = 0 .$$

However, if using them as complex numbers, only one of the above three complex forms must be exclusively used. Otherwise, when mixing them, any algebraic expression generally becomes quaternionic, where two or more of q_1, q_2, q_3 are non-zero, and manipulation of them must adhere to the non-commutative multiplication rules of quaternions, not complex numbers, which do commute. One reason to treat quaternions as complex numbers is for the purposes of checking that any quaternion algebra reverts to the algebra of complex numbers when the quaternion is simplified to one (and only one) basis, i.e. one of i, j, k but never two or more. Other than this, it would seem a futile use of quaternions if all they are used for is in their simplified, complex form.

(9-3) Multiplication

To see how quaternions multiply, a second quaternion $\mathbf{r}$ is defined similar to $\mathbf{q}$ for four, arbitrary real numbers r_0, r_1, r_2, r_3 by

(9.14) $\mathbf{r} = r_0 + r_1 i + r_2 j + r_3 k \,, \; r_0, r_1, r_2, r_3 \in \mathbb{R}$,

Multiplication is then simply a case of expanding the following bracketed product using the multiplication rules (9.1)

(9.15) $\mathbf{rq} = (r_0 + r_1 i + r_2 j + r_3 k)(q_0 + q_1 i + q_2 j + q_3 k)$.

Partially expanding this in terms of scalar and vector terms gives

(9.16) $$\mathbf{rq} = r_0 q_0 + r_0 (q_1 i + q_2 j + q_3 k) + (r_1 i + r_2 j + r_3 k) q_0 + (r_1 i + r_2 j + r_3 k)(q_1 i + q_2 j + q_3 k)$$

and then expanding a bit further to give

(9.17) $$\begin{aligned}\mathbf{rq} &= r_0q_0 + r_0q_1i + r_0q_2j + r_0q_3k + r_1q_0i + r_2q_0j + r_3q_0k \\ &+ r_1i(q_1i + q_2j + q_3k) \\ &+ r_2j(q_1i + q_2j + q_3k) \\ &+ r_3k(q_1i + q_2j + q_3k).\end{aligned}$$

The last three bracketed terms expand as follows:

(9.18)

$$\begin{aligned} r_1i(q_1i + q_2j + q_3k) &= r_1q_1i^2 + r_1q_2ij + r_1q_3ik \\ r_2j(q_1i + q_2j + q_3k) &= r_2q_1ji + r_2q_2j^2 + r_2q_3jk \\ r_3k(q_1i + q_2j + q_3k) &= r_3q_1ki + r_3q_2kj + r_3q_3k^2, \end{aligned}$$

and using the relations (9.1) these become

(9.19)

$$\begin{aligned} r_1i(q_1i + q_2j + q_3k) &= -r_1q_1 + r_1q_2k - r_1q_3j \\ r_2j(q_1i + q_2j + q_3k) &= -r_2q_1k - r_2q_2 + r_2q_3i \\ r_3k(q_1i + q_2j + q_3k) &= r_3q_1j - r_3q_2i - r_3q_3. \end{aligned}$$

Finally, collating these terms into their scalar and vector components, the quaternion product is thus

(9.20) $$\begin{aligned}\mathbf{rq} &= r_0q_0 - (r_1q_1 + r_2q_2 + r_3q_3) \\ &+ (r_0q_1 + r_1q_0 + r_2q_3 - r_3q_2)i \\ &+ (r_0q_2 + r_2q_0 + r_3q_1 - r_1q_3)j \\ &+ (r_0q_3 + r_3q_0 + r_1q_2 - r_2q_1)k.\end{aligned}$$

That was a rather long winded process that, fortunately, need not be repeated. Firstly, given readers of this book are assumed familiar with the inner (or *dot*) and vector cross product, then the quaternion product is evaluated more simply using the scalar/vector forms of the quaternions $\mathbf{q}$ and $\mathbf{r}$, where $\mathbf{r}$ is written in vector form as for $\mathbf{q}$ (9.7), i.e.

(9.21) $\mathbf{r} = r_0 + \underline{\mathbf{r}}, \ \underline{\mathbf{r}} = r_1i + r_2j + r_3k,$

with the column/row vector component

(9.22) $\underline{\mathbf{r}} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \ \underline{\mathbf{r}}^T = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix},$

and the full four-element quaternion written as

(9.23) $\mathbf{r} = \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}$ or $\mathbf{r} = \begin{pmatrix} r_0 \\ \underline{\mathbf{r}} \end{pmatrix}$, $\mathbf{r}^T = \begin{pmatrix} r_0 & \underline{\mathbf{r}}^T \end{pmatrix}$.

Using the vector definitions of $\mathbf{q}$ (9.7) and $\mathbf{r}$ (9.21), then the product $\mathbf{rq}$ can also be written as follows, with the inner product ($\underline{\mathbf{r}} \cdot \underline{\mathbf{q}}$) and cross product ($\underline{\mathbf{r}} \times \underline{\mathbf{q}}$) terms expanded upon next:

(9.24) $\mathbf{rq} = r_0 q_0 - \underline{\mathbf{r}} \cdot \underline{\mathbf{q}} + r_0 \underline{\mathbf{q}} + q_0 \underline{\mathbf{r}} + \underline{\mathbf{r}} \times \underline{\mathbf{q}}$.

Note that in URMT the inner vector product $\underline{\mathbf{r}} \cdot \underline{\mathbf{q}}$ is often written in the vector multiplication form $\underline{\mathbf{r}}^T \underline{\mathbf{q}}$, i.e.

(9.25) $\underline{\mathbf{r}} \cdot \underline{\mathbf{q}} = \underline{\mathbf{r}}^T \underline{\mathbf{q}} = \begin{pmatrix} r_1 & r_2 & r_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = (r_1 q_1 + r_2 q_2 + r_3 q_3) = \underline{\mathbf{q}}^T \underline{\mathbf{r}} = \underline{\mathbf{q}} \cdot \underline{\mathbf{r}}$,

so that the above product $\mathbf{rq}$ (9.24) is re-written as

(9.26) $\mathbf{rq} = r_0 q_0 - \underline{\mathbf{r}}^T \underline{\mathbf{q}} + r_0 \underline{\mathbf{q}} + q_0 \underline{\mathbf{r}} + \underline{\mathbf{r}} \times \underline{\mathbf{q}}$.

This transpose form $\underline{\mathbf{r}}^T \underline{\mathbf{q}}$ of the inner product $\underline{\mathbf{r}} \cdot \underline{\mathbf{q}}$ is preferred in URMT because, in matrix algebra, the product of a row vector with a column vector, in that order, automatically gives a scalar result. Furthermore, meaningful physical scalar quantities in URMT, e.g. observables such as energy, are invariably formed from such an inner product - witness the DCE (F10), which is given as the inner product between a zero, row eigenvector and its equivalent column vector.

The vector cross product $\underline{\mathbf{r}} \times \underline{\mathbf{q}}$ is often written in the, hopefully familiar, matrix determinant form as

(9.27) $\underline{\mathbf{r}} \times \underline{\mathbf{q}} = \begin{vmatrix} i & j & k \\ r_1 & r_2 & r_3 \\ q_1 & q_2 & q_3 \end{vmatrix} = (r_2 q_3 - r_3 q_2) i + (r_3 q_1 - r_1 q_3) j + (r_1 q_2 - r_2 q_1) k$.

Later, when using the matrix representation form of a quaternion, concepts of inner and cross vector products can be dispensed with, and the multiplication is then just that of matrix multiplication. However, beforehand, there are still several standard quaternion concepts to be discussed.

(9-4) Non-commutativity

Quaternions do not, in general, commute because the vector cross product (9.27) is non-commutative, i.e.

(9.28) $\underline{\mathbf{r}} \times \underline{\mathbf{q}} = -\underline{\mathbf{q}} \times \underline{\mathbf{r}}$.

For later reference, this result also simply shows that the cross product of a vector with itself is zero since substituting $\underline{\mathbf{q}}$ for $\underline{\mathbf{r}}$ gives

(9.29) $\mathbf{q}\times\underline{\mathbf{q}} = -\underline{\mathbf{q}}\times\mathbf{q} \Rightarrow \underline{\mathbf{q}}\times\mathbf{q} = 0$.

Looking at the quaternion product $\mathbf{rq}$ (9.26), then each of the terms on the right commutes barring the last, cross product. Starting with the scalar part $r_0 q_0$, the inner product $\underline{\mathbf{r}}^T\underline{\mathbf{q}}$ and the scalar multiples, $r_0\underline{\mathbf{q}}$ and $q_0\underline{\mathbf{r}}$, all commute. That is, $r_0 q_0 = q_0 r_0$ since real numbers commute, $\underline{\mathbf{r}}^T\underline{\mathbf{q}} = \underline{\mathbf{q}}^T\underline{\mathbf{r}}$ since the inner product commutes (to give a real, scalar result), and multiplication of a quaternion by a real number is also commutative (9.4). Thus, it can be deduced that the product $\mathbf{qr}$ only differentiates itself from $\mathbf{rq}$ by a change in sign of the cross product, as per (9.28), to give

(9.30) $\mathbf{qr} = r_0 q_0 - \underline{\mathbf{r}}^T\underline{\mathbf{q}} + r_0\underline{\mathbf{q}} + q_0\underline{\mathbf{r}} - \underline{\mathbf{r}}\times\underline{\mathbf{q}}$.

The difference in the two quaternion products $\mathbf{rq}$ and $\mathbf{qr}$ is therefore

(9.31) $\mathbf{rq} - \mathbf{qr} = 2\underline{\mathbf{r}}\times\underline{\mathbf{q}} = 2\begin{pmatrix} 0 \\ \underline{\mathbf{r}}\times\underline{\mathbf{q}} \end{pmatrix}$,

which has no real, scalar part involving r_0 and/or q_0.

By setting q_0 and r_0 to zero in (9.30), and using $\underline{\mathbf{r}}^T\underline{\mathbf{q}} = \underline{\mathbf{r}}\cdot\underline{\mathbf{q}}$ (9.25), the product of two pure vector quaternions $\underline{\mathbf{q}}$ and $\underline{\mathbf{r}}$ is

(9.32) $\underline{\mathbf{q}}\underline{\mathbf{r}} = -\underline{\mathbf{r}}\cdot\underline{\mathbf{q}} - \underline{\mathbf{r}}\times\underline{\mathbf{q}} = -\begin{pmatrix} \underline{\mathbf{r}}\cdot\underline{\mathbf{q}} \\ \underline{\mathbf{r}}\times\underline{\mathbf{q}} \end{pmatrix}$,

and likewise

(9.33) $\underline{\mathbf{r}}\underline{\mathbf{q}} = -\underline{\mathbf{r}}\cdot\underline{\mathbf{q}} + \underline{\mathbf{r}}\times\underline{\mathbf{q}} = \begin{pmatrix} -\underline{\mathbf{r}}\cdot\underline{\mathbf{q}} \\ \underline{\mathbf{r}}\times\underline{\mathbf{q}} \end{pmatrix}$.

These two results mean that the inner product and cross product can both be defined solely in terms of a quaternion vector product, i.e.

(9.34) $\underline{\mathbf{r}}\cdot\underline{\mathbf{q}} = -\frac{1}{2}\left(\underline{\mathbf{r}}\underline{\mathbf{q}} + \underline{\mathbf{q}}\underline{\mathbf{r}}\right)$

(9.35) $\underline{\mathbf{r}}\times\underline{\mathbf{q}} = \frac{1}{2}\left(\underline{\mathbf{r}}\underline{\mathbf{q}} - \underline{\mathbf{q}}\underline{\mathbf{r}}\right)$.

This result has similarities to the 'geometric product', which expands a product of two vectors as the sum of an inner and exterior (or wedge) product (I40). Without going into any more unnecessary detail (here), the reader is referred to [14] for more information. Lastly on this matter, although the similarity between quaternions and geometric algebra is not required or pursued further herein, the connections between quaternions, URMT and geometric algebra is likely to reappear in the future.

The square of a quaternion is easily evaluated by substituting $\mathbf{q}$ for $\mathbf{r}$ in (9.30) to give, in full,

(9.36) $\mathbf{q}^2 = q_0^2 - \underline{\mathbf{q}}^T \underline{\mathbf{q}} + 2q_0 \underline{\mathbf{q}} - \underline{\mathbf{q}} \times \underline{\mathbf{q}}$.

Since the last term on the right is zero by the vector cross product relation $\underline{\mathbf{q}} \times \underline{\mathbf{q}} = 0$ (9.29), and using the inner product $\underline{\mathbf{q}} \cdot \underline{\mathbf{q}} = |\underline{\mathbf{q}}|^2$, (9.40) below, then the square of quaternion $\mathbf{q}$ is

(9.37) $\mathbf{q}^2 = q_0^2 - |\underline{\mathbf{q}}|^2 + 2q_0 \underline{\mathbf{q}}$,

alternatively written in block vector form as

(9.38) $\mathbf{q}^2 = \begin{pmatrix} q_0^2 - |\underline{\mathbf{q}}|^2 \\ 2q_0 \underline{\mathbf{q}} \end{pmatrix}$.

From this, the vector component of the quaternion squared, i.e. $\underline{\mathbf{q}}^2$, is easily obtained by setting $q_0 = 0$ to give

(9.39) $\underline{\mathbf{q}}\underline{\mathbf{q}} = \underline{\mathbf{q}}^2 = -|\underline{\mathbf{q}}|^2$ or $\underline{\mathbf{q}}^2 = \begin{pmatrix} -|\underline{\mathbf{q}}|^2 \\ 0 \end{pmatrix}$.

Thus $\underline{\mathbf{q}}^2$ comprises only a real, scalar part with no vector component.

Note that the inner product of the vector component $\underline{\mathbf{q}}$ with itself, i.e. $\underline{\mathbf{q}} \cdot \underline{\mathbf{q}}$ ($\equiv \underline{\mathbf{q}}^T \underline{\mathbf{q}}$), is the squared magnitude of $\underline{\mathbf{q}}$, and the negative of the quaternion vector product $\underline{\mathbf{q}}\underline{\mathbf{q}}$. The inner product is written in its myriad, notational forms as

(9.40) $\underline{\mathbf{q}} \cdot \underline{\mathbf{q}} = \underline{\mathbf{q}}^T \underline{\mathbf{q}} = \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \left(q_1^2 + q_2^2 + q_3^2\right) = |\underline{\mathbf{q}}|^2$,

$|\underline{\mathbf{q}}| = \sqrt{\underline{\mathbf{q}} \cdot \underline{\mathbf{q}}} \geq 0$.

Caution. It is very easy to confuse these two products $\underline{\mathbf{q}} \cdot \underline{\mathbf{q}}$ and $\underline{\mathbf{q}}^2$. However, the inner product is only a product between components, and takes no account of the multiplication rules (9.1) of the quaternion bases i, j, k, whereas $\underline{\mathbf{q}}^2$ does take account of these rules, i.e., in full,

(9.41) $\underline{\mathbf{q}}^2 = (q_1 i + q_2 j + q_3 k)(q_1 i + q_2 j + q_3 k) = -\left(q_1^2 + q_2^2 + q_3^2\right) = -|\underline{\mathbf{q}}|^2$

(9-5) Associativity and Distributivity

Quaternion multiplication is associative, i.e. if $\mathbf{q}$, $\mathbf{r}$ and $\mathbf{s}$ are three quaternions then

(9.42) $\mathbf{q}(\mathbf{rs}) = (\mathbf{qr})\mathbf{s}$.

Quaternion multiplication is also distributive over addition, i.e.

(9.43) $\mathbf{q}(\mathbf{r}+\mathbf{s}) = \mathbf{qr}+\mathbf{qs}$

Of course, the order of execution must be observed, i.e.

(9.44) $\mathbf{q}(\mathbf{r}+\mathbf{s}) = \mathbf{qr}+\mathbf{qs} \neq (\mathbf{r}+\mathbf{s})\mathbf{q} = \mathbf{rq}+\mathbf{sq}$

In fact, quaternions obey all the usual algebraic properties of real numbers and complex numbers, including division (see later), barring commutativity.

Readers may be familiar with one further type of number, namely Octonions, which have one real scalar and seven complex components, the latter forming the 'vector' part. These octonions are neither commutative nor associative. URMT has no current use for them and they are not used further - see [8] for some more information.

(9-6) Miscellaneous Products

The product $\mathbf{q}\underline{\mathbf{q}}$ evaluates as follows, using $\mathbf{q} = q_0 + \underline{\mathbf{q}}$ (9.7) and the above result $\underline{\mathbf{q}}^2 = -|\underline{\mathbf{q}}|^2$ (9.39):

(9.45) $\mathbf{q}\underline{\mathbf{q}} = (q_0 + \underline{\mathbf{q}})\underline{\mathbf{q}} = q_0\underline{\mathbf{q}} - |\underline{\mathbf{q}}|^2 = \begin{pmatrix} -|\underline{\mathbf{q}}|^2 \\ q_0\underline{\mathbf{q}} \end{pmatrix}$.

The product $\mathbf{q}\underline{\mathbf{q}}$ is identical to $\underline{\mathbf{q}}\mathbf{q}$ since real numbers commute with quaternions, i.e. $\underline{\mathbf{q}}q_0 = q_0\underline{\mathbf{q}}$ in the following:

(9.46) $\underline{\mathbf{q}}\mathbf{q} = \underline{\mathbf{q}}(q_0 + \underline{\mathbf{q}}) = \underline{\mathbf{q}}q_0 - |\underline{\mathbf{q}}|^2 \Rightarrow \mathbf{q}\underline{\mathbf{q}} = \underline{\mathbf{q}}\mathbf{q}$.

(9-7) Quaternion Conjugate and Magnitude

Just as per a complex number, a quaternion has a conjugate $\mathbf{q}^*$ defined simply as

(9.47) $\mathbf{q}^* = q_0 - \underline{\mathbf{q}} = \begin{pmatrix} q_0 \\ -\underline{\mathbf{q}} \end{pmatrix}$,

and the squared magnitude, also known as the norm (see note (9.53) below) or length of the quaternion, is also defined in the same way as per a complex number, i.e. the product of the quaternion and its conjugate:

(9.48) $|\mathbf{q}|^2 = \mathbf{q}^*\mathbf{q}$.

The product $\mathbf{q}^*\mathbf{q}$ is quickly evaluated by substituting for $\mathbf{q}^*$ in place of $\mathbf{r}$ in the product $\mathbf{rq}$ (9.24) to give, using $\underline{\mathbf{q}} \times \underline{\mathbf{q}} = 0$ (9.29),

(9.49) $\mathbf{q}^*\mathbf{q} = q_0^2 + \underline{\mathbf{q}} \cdot \underline{\mathbf{q}}$,

and using (9.40) then the square of the quaternion magnitude is thus just the sum of the squares of the components, as for a complex number, i.e.

(9.50) $\mathbf{q}^*\mathbf{q} = |\mathbf{q}|^2 = q_0^2 + |\underline{\mathbf{q}}|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2$, $|\mathbf{q}|, |\underline{\mathbf{q}}| \geq 0$.

Since the product $\mathbf{q}^*\mathbf{q}$ is the real scalar $|\mathbf{q}|^2$, then the product $\mathbf{qq}^*$ is the same as $\mathbf{q}^*\mathbf{q}$, i.e.

(9.51) $\mathbf{qq}^* = (\mathbf{q}^*\mathbf{q})^* = |\mathbf{q}|^2$,

and so a quaternion commutes with its conjugate, i.e.

(9.52) $\mathbf{qq}^* = \mathbf{q}^*\mathbf{q}$.

(9.53)
Note that whilst the two terms 'norm' and 'magnitude' are used interchangeably when talking about quaternions, they are not the same quantities in URMT. Although the URMT magnitude definition (I7) is in agreement with common usage, the norm (I8) in URMT is not the same as the magnitude, and the norm of all eigenvectors (to non-zero eigenvalues) is, in fact, zero. Note too that the URMT magnitude is also zero for complex eigenvectors, which may have one or more complex elements. Such eigenvectors were first introduced in [3],10, and reviewed in Section (1-3) in the case of URM3. Also note that there is no over-arching reason why any of the quaternion components q_0, q_1, q_2, q_3 themselves cannot be complex numbers, but such 'complexification' is not required here.

Since the components of a quaternion are all real numbers herein, at least one of which is non-zero (when embedded as URMT eigenvectors AVE I (1.1)), then the magnitude is always non-zero, which leads to a legitimate concept of division, exactly as per complex numbers (once again), and discussed in the next section. In fact, by convention, the magnitude $|\mathbf{q}|$ is defined as positive, as for all magnitude quantities (1.2), and hence too $-|\mathbf{q}|$ is always less than zero by the same convention. This latter point is mentioned because such negative quantities are used in URMT's AVE I and II eigenvector treatment - see the eigenvector $\mathbf{X}_-$ (1.25), where $|\mathbf{q}| \sim |\mathbf{X}|$.

Lastly on the subject of magnitude, in the application of quaternions to rotations, Sections (11) and (12), the quaternion magnitude is usually constrained to unity. However, there is no overriding reason for this in general quaternion algebra.

(9-8) The Conjugate of a Product

The conjugate of the quaternion product $\mathbf{rq}$, denoted by $(\mathbf{rq})^*$, expands as

(9.54) $(\mathbf{rq})^* = \mathbf{q}^*\mathbf{r}^*$,

which is similar to the matrix transpose rule $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

This is easily shown by taking the conjugate of the individual terms of the product $\mathbf{rq}$ (9.24), i.e.

(9.55) $(\mathbf{rq})^* = (r_0 q_0 - \underline{\mathbf{r}}^T \underline{\mathbf{q}})^* + (r_0 \underline{\mathbf{q}})^* + (q_0 \underline{\mathbf{r}})^* + (\underline{\mathbf{r}} \times \underline{\mathbf{q}})^*$.

Since the conjugate of a real scalar is the same scalar, with no sign reversal, then the conjugate of the scalar component of the quaternion is unchanged by conjugation, i.e.

(9.56) $(r_0 q_0 - \underline{\mathbf{r}}^T \underline{\mathbf{q}})^* = r_0 q_0 - \underline{\mathbf{r}}^T \underline{\mathbf{q}}$,

similarly, since $r_0^* = r_0$ and $q_0^* = q_0$, then

$$(r_0 \underline{\mathbf{q}})^* = r_0 \underline{\mathbf{q}}^*,\ (q_0 \underline{\mathbf{r}})^* = q_0 \underline{\mathbf{r}}^*.$$

Using the following conjugations:

(9.57) $\underline{\mathbf{q}}^* = -\underline{\mathbf{q}},\ \underline{\mathbf{r}}^* = -\underline{\mathbf{r}},\ (\underline{\mathbf{r}} \times \underline{\mathbf{q}})^* = -(\underline{\mathbf{r}} \times \underline{\mathbf{q}}) = (\underline{\mathbf{q}} \times \underline{\mathbf{r}})$,

then $(\mathbf{rq})^*$ becomes

(9.58) $(\mathbf{rq})^* = r_0 q_0 - \underline{\mathbf{r}}^T \underline{\mathbf{q}} - r_0 \underline{\mathbf{q}} - q_0 \underline{\mathbf{r}} + \underline{\mathbf{q}} \times \underline{\mathbf{r}}$.

The product $\mathbf{q}^*\mathbf{r}^*$ is evaluated as

(9.59) $\mathbf{q}^*\mathbf{r}^* = q_0 r_0 - (\underline{-\mathbf{q}^T})(-\underline{\mathbf{r}}) - r_0 \underline{\mathbf{q}} - q_0 \underline{\mathbf{r}} + (-\underline{\mathbf{q}}) \times (-\underline{\mathbf{r}})$,

and since $(\underline{-\mathbf{q}^T})(-\underline{\mathbf{r}}) = \underline{\mathbf{q}} \cdot \underline{\mathbf{r}} = \underline{\mathbf{r}} \cdot \underline{\mathbf{q}} = \underline{\mathbf{r}}^T \underline{\mathbf{q}}$ and $(-\underline{\mathbf{q}}) \times (-\underline{\mathbf{r}}) = \underline{\mathbf{q}} \times \underline{\mathbf{r}}$ then $\mathbf{q}^*\mathbf{r}^*$ is seen to be identical to $(\mathbf{rq})^*$.

Using conjugate forms, it can easily be shown (to within a sign) that the magnitude of the product of two quaternions is the product of the magnitude of each quaternion, as for complex numbers, i.e.

(9.60) $|\mathbf{rq}| = |\mathbf{r}||\mathbf{q}|$.

From the squared magnitude definition (9.50), using (9.54) and associativity (9.42), then

(9.61) $|\mathbf{rq}|^2 = (\mathbf{rq})^*(\mathbf{rq}) = \left(\mathbf{q}^*\mathbf{r}^*\right)(\mathbf{rq}) = \mathbf{q}^*\left(\mathbf{r}^*\mathbf{r}\right)\mathbf{q}$

$= \mathbf{q}^*|\mathbf{r}|^2\mathbf{q} = |\mathbf{r}|^2\left(\mathbf{q}^*\mathbf{q}\right) = |\mathbf{r}|^2|\mathbf{q}|^2 .$

Taking the positive square root thus gives (9.60). Of course, the negative square root could also be taken, but magnitudes are defined here as zero or positive (9.40). In fact, URMT only uses non-zero, positive magnitudes, which means quaternion division (next) is always well-defined.

(9-9) Quaternion division

The reciprocal of a non-zero magnitude quaternion $\mathbf{q}$ is obtained as per a complex reciprocal by multiplying top and bottom by the conjugate, thereby reducing the denominator to a real number, i.e.

(9.62) $\mathbf{q}^{-1} = \dfrac{1}{\mathbf{q}} = \dfrac{\mathbf{q}^*}{\mathbf{q}\mathbf{q}^*} = \left(\dfrac{q_0 - \underline{\mathbf{q}}}{|\mathbf{q}|^2}\right)$, $\mathbf{q}\mathbf{q}^* = |\mathbf{q}|^2$ (9.51),

$\mathbf{q}^* = q_0 - \underline{\mathbf{q}}$ (9.47), $|\mathbf{q}|^2 > 0$.

Thus, the reciprocal of $\mathbf{q}$ is just the conjugate quaternion divided by the squared magnitude.

Armed with the reciprocal, it might then seem straightforward to divide a quaternion $\mathbf{r}$ by a non-zero $\mathbf{q}$ simply by multiplying $\mathbf{r}$ by the reciprocal $\mathbf{q}^{-1}$, i.e.

(9.63) $\dfrac{\mathbf{r}}{\mathbf{q}} = \mathbf{r}\mathbf{q}^{-1}$.

However, note that this is multiplication of $\mathbf{r}$ by $\mathbf{q}^{-1}$ on the right. Since quaternion multiplication is not commutative, it should not be expected to give the same results as multiplying by the reciprocal on the left, i.e.

(9.64) $\mathbf{r}\mathbf{q}^{-1} \neq \mathbf{q}^{-1}\mathbf{r}$.

Thus, using the result $\mathbf{q}^{-1}$ (9.62), the two possible divisions become

(9.65) $\mathbf{r}\mathbf{q}^{-1} = \dfrac{1}{|\mathbf{q}|^2}\left(\mathbf{r}q_0 - \mathbf{r}\underline{\mathbf{q}}\right)$

(9.66) $\mathbf{q}^{-1}\mathbf{r} = \dfrac{1}{|\mathbf{q}|^2}\left(q_0\mathbf{r} - \underline{\mathbf{q}}\mathbf{r}\right)$,

and the difference in the two divisions is therefore

(9.67) $\mathbf{r}\mathbf{q}^{-1} - \mathbf{q}^{-1}\mathbf{r} = \dfrac{1}{|\mathbf{q}|^2}\left(\underline{\mathbf{q}}\mathbf{r} - \mathbf{r}\underline{\mathbf{q}}\right)$.

Using the quaternion multiplication rule (9.24) for $\mathbf{q} = \underline{\mathbf{q}}$ and $q_0 = 0$, the two numerator terms are

(9.68) $\underline{\mathbf{q}}\mathbf{r} = -\underline{\mathbf{q}} \cdot \underline{\mathbf{r}} + r_0 \underline{\mathbf{q}} + \underline{\mathbf{q}} \times \underline{\mathbf{r}}$

(9.69) $\mathbf{r}\underline{\mathbf{q}} = -\underline{\mathbf{r}} \cdot \underline{\mathbf{q}} + r_0 \underline{\mathbf{q}} + \underline{\mathbf{r}} \times \underline{\mathbf{q}}$.

and the difference in the products $\underline{\mathbf{q}}\mathbf{r}$ and $\mathbf{r}\underline{\mathbf{q}}$ is therefore

(9.70) $\underline{\mathbf{q}}\mathbf{r} - \mathbf{r}\underline{\mathbf{q}} = \mathbf{q} \times \underline{\mathbf{r}} - \underline{\mathbf{r}} \times \underline{\mathbf{q}}$.

Thus, since $\underline{\mathbf{r}} \times \underline{\mathbf{q}} = -\,\mathbf{q} \times \underline{\mathbf{r}}$, the difference (9.67) is given by

(9.71) $\mathbf{r}\mathbf{q}^{-1} - \mathbf{q}^{-1}\mathbf{r} = \frac{2}{|\mathbf{q}|^2}\left(\underline{\mathbf{q}} \times \underline{\mathbf{r}}\right)$,

which is a pure vector quaternion with no scalar part.

This completes the basic algebraic properties of quaternions required herein. The next section shows how a matrix can be used to represent a quaternion.

10 The Quaternion Representation Matrix

This section details the 4x4 matrix representation of a quaternion. Such a representation is very important to URMT, specifically the work within this book, because it is can be treated as per a 4x4 skew-symmetric matrix using URMT's method of AVE I, Section (1), and is also fundamental to URMT's treatment of rotations in Sections (13) and (14) on spin and rigid body dynamics.

(10-1) Definition

A quaternion $\mathbf{q}$ with a real, scalar component q_0 and vector component $\underline{\mathbf{q}}$, as first defined in in the previous Section, reproduced below,

$$\mathbf{q} = \begin{pmatrix} q_0 \\ \underline{\mathbf{q}} \end{pmatrix} \text{ (9.10)}, \quad \underline{\mathbf{q}} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \text{ (9.8)},$$

has an equivalent, 4x4, matrix representation, denoted by $\mathbf{Q_q}$ and defined as

$$\text{(10.1)} \quad \mathbf{Q_q} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \sim \mathbf{q} = \begin{pmatrix} q_0 \\ \underline{\mathbf{q}} \end{pmatrix}.$$

If $\mathbf{r}$ is a second quaternion

$$\text{(10.2)} \quad \mathbf{r} = \begin{pmatrix} r_0 \\ \underline{\mathbf{r}} \end{pmatrix}, \quad \underline{\mathbf{r}} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix},$$

then the quaternion product $\mathbf{qr}$ is equivalent to the following matrix-vector product (linear transformation):

$$\text{(10.3)} \quad \mathbf{qr} = \mathbf{Q_q}\mathbf{r} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

and hence matrix $\mathbf{Q_q}$ is a representation of the quaternion $\mathbf{q}$ under multiplication.

The representation matrix $\mathbf{Q_q}$ is nicer split into two matrix components, one for the scalar q_0 and another for the vector components q_1, q_2, q_3 as follows, where $\mathbf{I}_4$ is the identity matrix:

(10.4) $$\mathbf{Q_q} = q_0\mathbf{I}_4 + \begin{pmatrix} 0 & -q_1 & -q_2 & -q_3 \\ q_1 & 0 & -q_3 & q_2 \\ q_2 & q_3 & 0 & -q_1 \\ q_3 & -q_2 & q_1 & 0 \end{pmatrix}.$$

This split form is better written in the more concise, block matrix form in terms of q_0 and vector $\underline{\mathbf{q}}$ as

(10.5) $$\mathbf{Q_q} = q_0\mathbf{I}_4 + \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix},$$

where the bottom right, 3x3 sub-matrix is the cross product, matrix operator '$\underline{\mathbf{q}}\times$' defined as

(10.6) $$\underline{\mathbf{q}}\times = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix},$$

and discussed again later, sub-section (10-3).

The block matrix form of $\mathbf{Q_q}$ (10.5) can also be written as follows:

(10.7) $$\mathbf{Q_q} = \begin{pmatrix} q_0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & q_0\mathbf{I}_3 + \underline{\mathbf{q}}\times \end{pmatrix},$$

where it is clear that the left column is the quaternion $\mathbf{q}$ (9.10). This fact is useful later when extracting the quaternion result from a product of two quaternion representation matrices, see (10.54) further below.

By defining the two, 4x4 matrices on the right of (10.5) as

(10.8) $$\mathbf{Q}_0 = q_0\mathbf{I}_4 \sim \begin{pmatrix} q_0 \\ 0 \end{pmatrix}$$

(10.9) $$\mathbf{Q}_{\underline{\mathbf{q}}} = \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix} \sim \begin{pmatrix} 0 \\ \underline{\mathbf{q}} \end{pmatrix},$$

then the quaternion representation matrix is written as the matrix sum

(10.10) $$\mathbf{Q_q} = \mathbf{Q}_0 + \mathbf{Q}_{\underline{\mathbf{q}}} \sim \begin{pmatrix} q_0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \underline{\mathbf{q}} \end{pmatrix} = \begin{pmatrix} q_0 \\ \underline{\mathbf{q}} \end{pmatrix}.$$

Each matrix is a quaternion representation matrix in its own right, except that $\mathbf{Q}_0$ is a representation of a scalar quaternion comprising only the real number q_0, whilst $\mathbf{Q}_{\underline{\mathbf{q}}}$ comprises only the vector $\underline{\mathbf{q}}$.

The matrix $\mathbf{Q}_{\underline{\mathbf{q}}}$ is skew-symmetric, i.e.

(10.11) $\mathbf{Q}_{\underline{\mathbf{q}}}^T = -\mathbf{Q}_{\underline{\mathbf{q}}}$,

written in block form as

(10.12) $\mathbf{Q}_{\underline{\mathbf{q}}}^T = \begin{pmatrix} 0 & \underline{\mathbf{q}}^T \\ -\underline{\mathbf{q}} & -\underline{\mathbf{q}}\times \end{pmatrix}$.

Comparing this with (10.9) it can be seen that $\mathbf{Q}_{\underline{\mathbf{q}}}^T$ is just the quaternion representation matrix for $-\underline{\mathbf{q}}$, i.e.

(10.13) $\mathbf{Q}_{\underline{\mathbf{q}}}^T = -\mathbf{Q}_{\underline{\mathbf{q}}} = \mathbf{Q}_{-\underline{\mathbf{q}}} \sim \begin{pmatrix} 0 \\ -\underline{\mathbf{q}} \end{pmatrix}$.

Conversely, because $\mathbf{Q}_0$ is symmetric then

$$\mathbf{Q}_0^T = \mathbf{Q}_0 \sim \begin{pmatrix} q_0 \\ 0 \end{pmatrix}.$$

Taking the transpose of $\mathbf{Q}_{\mathbf{q}}$ (10.10) and using the above results for $\mathbf{Q}_{\underline{\mathbf{q}}}^T$ and $\mathbf{Q}_0^T$ gives

(10.14) $\mathbf{Q}_{\mathbf{q}}^T = \mathbf{Q}_0^T + \mathbf{Q}_{\underline{\mathbf{q}}}^T = \mathbf{Q}_0 - \mathbf{Q}_{\underline{\mathbf{q}}} = \mathbf{Q}_0 + \mathbf{Q}_{-\underline{\mathbf{q}}}$.

However $\mathbf{Q}_0 + \mathbf{Q}_{-\underline{\mathbf{q}}}$ is just the quaternion representation matrix for the quaternion $q_0 - \underline{\mathbf{q}}$, which itself is just the conjugate quaternion $\mathbf{q}^*$ (9.47), i.e.

$$\mathbf{Q}_0 + \mathbf{Q}_{-\underline{\mathbf{q}}} \sim \begin{pmatrix} q_0 \\ -\underline{\mathbf{q}} \end{pmatrix} = \mathbf{q}^* \text{ (9.47).}$$

So, finally, the transpose $\mathbf{Q}_{\mathbf{q}}^T$ is thus the same as the conjugate $\mathbf{Q}_{\mathbf{q}}^*$, i.e.

(10.15) $\mathbf{Q}_{\mathbf{q}}^T = \mathbf{Q}_0 + \mathbf{Q}_{-\underline{\mathbf{q}}} = \mathbf{Q}_{\mathbf{q}}^*$,

$$\mathbf{Q}_{\mathbf{q}}^* \sim \begin{pmatrix} q_0 \\ -\underline{\mathbf{q}} \end{pmatrix} = \mathbf{q}^*.$$

Before detailing $\mathbf{Q}_{\mathbf{q}}$ further, it is noted that matrix $\mathbf{Q}_{\underline{\mathbf{q}}}$ (10.9) is identical in form to a URM4 skew matrix, i.e. $\mathbf{A}_4^S$ Skew form 2 (1.83), which also possesses an all-zero lead diagonal, and has therefore already been studied in URMT under the subject of URM4 skew matrices, Section (1-4) - more details are supplied further below, see (10.21) onward.

(10-2) Eigenvalues and Eigenvectors

Although $\mathbf{Q}_{\underline{\mathbf{q}}}$ (10.9) is only half the story, i.e. the full quaternion matrix $\mathbf{Q}_{\mathbf{q}}$ also involves $\mathbf{Q}_0$, this latter matrix is nothing more than a scaled multiple of the identity matrix $\mathbf{I}_4$ and, as such, the eigenvalues and eigenvectors of the representation matrix $\mathbf{Q}_{\mathbf{q}}$ are easily determined from those of $\mathbf{Q}_{\underline{\mathbf{q}}}$. This is because, if a vector $\mathbf{X}$ is an eigenvector of the matrix $\mathbf{Q}_{\underline{\mathbf{q}}}$, for eigenvalue λ, then it is an eigenvector of $\mathbf{Q}_{\mathbf{q}}$ for eigenvalue $q_0 + \lambda$. This is easily shown starting with the definition

(10.16) $\mathbf{Q}_{\underline{\mathbf{q}}}\mathbf{X} = \lambda\mathbf{X}$,

and since

(10.17) $\mathbf{Q}_0\mathbf{X} = q_0\mathbf{I}_4\mathbf{X} = q_0\mathbf{X}$,

then, by the definition of $\mathbf{Q}_{\mathbf{q}}$ (10.10) , $\mathbf{Q}_{\mathbf{q}}\mathbf{X}$ is given by

(10.18) $\mathbf{Q}_{\mathbf{q}}\mathbf{X} = (\mathbf{Q}_0 + \mathbf{Q}_{\underline{\mathbf{q}}})\mathbf{X} = \mathbf{Q}_0\mathbf{X} + \mathbf{Q}_{\underline{\mathbf{q}}}\mathbf{X}$.

Using (10.16) and (10.17) gives

(10.19) $\mathbf{Q}_0\mathbf{X} + \mathbf{Q}_{\underline{\mathbf{q}}}\mathbf{X} = q_0\mathbf{X} + \lambda\mathbf{X} = (q_0 + \lambda)\mathbf{X}$,

and thus it is seen that $\mathbf{X}$ is an also eigenvector of the matrix $\mathbf{Q}_{\mathbf{q}}$ for eigenvalue $q_0 + \lambda$, i.e.

(10.20) $\mathbf{Q}_{\mathbf{q}}\mathbf{X} = (q_0 + \lambda)\mathbf{X}$.

In other words, knowing the eigenvalues and eigenvectors of $\mathbf{Q}_{\underline{\mathbf{q}}}$ gives those of $\mathbf{Q}_{\mathbf{q}}$.

Summarising the above, whilst the quaternion representation matrix $\mathbf{Q}_{\mathbf{q}}$ is not, in itself, a form of URMT **A** matrix, it is, by virtue of the non-zero diagonal component matrix $\mathbf{Q}_0$, still simple enough that its eigenvalues and eigenvectors can be studied just by looking at the vector component matrix $\mathbf{Q}_{\underline{\mathbf{q}}}$. This matrix $\mathbf{Q}_{\underline{\mathbf{q}}}$ is a standard URM4, Skew form 2 matrix $\mathbf{A}_4^S$ (1.83), and therefore a solved problem by the method of AVE I, Section (1-4). Both matrices $\mathbf{Q}_{\underline{\mathbf{q}}}$ and $\mathbf{A}_4^S$ are reproduced below for comparison - with the eigenvector solution for $\mathbf{Q}_{\underline{\mathbf{q}}}$ given afterward.

(10.21) $\mathbf{Q}_{\underline{\mathbf{q}}} = \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix} \sim \mathbf{A}_4^S = \begin{pmatrix} 0 & -\mathbf{X}^T \\ \mathbf{X} & \mathbf{\Delta}_3^S \end{pmatrix}$ (1.83), Skew form 2, with the following equivalences:

(10.22) $\underline{\mathbf{q}} \sim \mathbf{X}$, $\underline{\mathbf{q}} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$, $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $q_1 \sim x$, $q_2 \sim y$, $q_3 \sim z$,

(10.23) $(\underline{\mathbf{q}}\times) \sim \mathbf{\Delta}_3^S$, $\underline{\mathbf{q}}\times = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}$, $\mathbf{\Delta}_3^S = \mathbf{X}\times = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$.

Note that the Skew form 1 (1.81) of $\mathbf{A}_4^S$ is not used here.

Using these equivalences, the full eigenvector solution for $\mathbf{Q}_{\underline{\mathbf{q}}}$ is given below in accordance with the URM4 Skew AVE solution, Section (1-4), (1.90) onward. Note that the URM3 eigenvectors $\mathbf{X}_{3i\pm}$ (eigenvectors of $\mathbf{\Delta}_3^S$), embedded in $\mathbf{X}_{4j\pm}$, are given afterward, followed by their definitions. The matrix $\mathbf{Q}_{\underline{\mathbf{q}}}$ has four eigenvalues, only two of which are unique, i.e. $\lambda = \pm i|\underline{\mathbf{q}}|$ repeated, with four distinct eigenvectors, $\mathbf{X}_{4i\pm}$ and $\mathbf{X}_{4j\pm}$:

(10.24)

$$\mathbf{X}_{4i\pm} = \begin{pmatrix} \pm i|\underline{\mathbf{q}}| \\ \underline{\mathbf{q}} \end{pmatrix}, \ \mathbf{Q}_{\underline{\mathbf{q}}}\mathbf{X}_{4i+} = \pm i|\underline{\mathbf{q}}|\mathbf{X}_{4i+}, \ \lambda = \pm i|\underline{\mathbf{q}}|$$

$$\mathbf{X}_{4j\pm} = \begin{pmatrix} 0 \\ \mathbf{X}_{3i\pm} \end{pmatrix}, \ \mathbf{Q}_{\underline{\mathbf{q}}}\mathbf{X}_{4j+} = \pm i|\underline{\mathbf{q}}|\mathbf{X}_{4j+}, \ \lambda = \pm i|\underline{\mathbf{q}}|$$

$$\mathbf{X}_{3i\pm} = \frac{q_1}{r}\underline{\mathbf{q}} \pm i\frac{|\underline{\mathbf{q}}|}{r}\begin{pmatrix} i|\underline{\mathbf{q}}| \\ q_3 \\ -q_2 \end{pmatrix}$$

$|\underline{\mathbf{q}}| = \sqrt{\underline{\mathbf{q}} \cdot \underline{\mathbf{q}}} > 0$ (9.40) see note below

$$r = \sqrt[+]{|\underline{\mathbf{q}}|^2 - q_1^2} = \sqrt[+]{q_2^2 + q_3^2} .$$

Note that the magnitude here is strictly greater than zero, rather than greater than or equal to zero as in (9.40). This is because the URMT *no singularity* rule (I34) is applied and the vector $\underline{\mathbf{q}}$ is therefore never zero, i.e. at least one component of $\underline{\mathbf{q}}$ is non-zero:

(10.25) $\underline{\mathbf{q}} \neq 0$, $(q_1, q_2, q_3) \neq (0,0,0)$, URMT *no singularity* (I34).

Thus, armed with the above eigenvector solution for $\mathbf{Q}_{\underline{\mathbf{q}}}$, for the four eigenvalues $\lambda_{\underline{\mathbf{q}}} \pm i|\underline{\mathbf{q}}|$ repeated, the eigenvalues of $\mathbf{Q}_{\mathbf{q}}$ are given in accordance with the earlier derivation (10.20) as

(10.26) $\lambda_{\mathbf{q}} = q_0 \pm i|\underline{\mathbf{q}}|$ repeated.

The eigenvectors also remain exactly the same as above (10.24), and using the above eigenvalues $\lambda_{\mathbf{q}}$, the eigenvector equations for the quaternion representation matrix $\mathbf{Q}_{\mathbf{q}}$ are thus

(10.27)

$$\mathbf{Q}_{\mathbf{q}}\mathbf{X}_{4i\pm} = (q_0 \pm i|\underline{\mathbf{q}}|)\mathbf{X}_{4i\pm},$$
$$\mathbf{Q}_{\mathbf{q}}\mathbf{X}_{4j\pm} = (q_0 \pm i|\underline{\mathbf{q}}|)\mathbf{X}_{4j\pm}.$$

(10-3) Some algebraic aspects of the 3x3 skew, sub-matrix

That the bottom right, 3x3 sub-matrix of $\mathbf{Q}_{\underline{\mathbf{q}}}$ (10.21) is the matrix representation of the cross product operator $\underline{\mathbf{q}}\times$ (10.6) can be seen by the action of $\underline{\mathbf{q}}\times$ on an arbitrary vector $\underline{\mathbf{r}}$, where the matrix vector product evaluates to

(10.28) $$\underline{\mathbf{q}}\times\underline{\mathbf{r}} = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -q_3r_2 + q_2r_3 \\ q_3r_1 - q_1r_3 \\ -q_2r_1 + q_1r_2 \end{pmatrix},$$

which is the same as the algebraic product given in determinant form - see $\underline{\mathbf{r}}\times\underline{\mathbf{q}}$ (9.27) and reverse the sign of all terms since $\underline{\mathbf{r}}\times\underline{\mathbf{q}} = -\underline{\mathbf{q}}\times\mathbf{r}$ (9.28).

In URMT, the cross product matrix form $\underline{\mathbf{q}}\times$ is known as a skew 'annihilator', $\mathbf{\Delta}^S$ (I23), as it annihilates its own vector, i.e. $\underline{\mathbf{q}}\times\underline{\mathbf{q}} = 0$ (9.29), expanded in full as

(10.29) $$\underline{\mathbf{q}}\times\underline{\mathbf{q}} = \begin{pmatrix} 0 & -q_3 & q_2 \\ q_3 & 0 & -q_1 \\ -q_2 & q_1 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = 0.$$

This annihilation also means that $\underline{\mathbf{q}}$ is a zero eigenvector (I18) of the matrix $\underline{\mathbf{q}}\times$.

Returning to the matrix vector product $\underline{\mathbf{q}}\times\underline{\mathbf{r}}$, it is also usefully re-written in the following form:

(10.30) $$\underline{\mathbf{q}}\times\underline{\mathbf{r}} = \begin{pmatrix} -q_3r_2 + q_2r_3 \\ q_3r_1 - q_1r_3 \\ -q_2r_1 + q_1r_2 \end{pmatrix} = \begin{pmatrix} 0q_1 + r_3q_2 - r_2q_3 \\ -r_3q_1 + 0q_2 + r_1q_3 \\ r_2q_1 - r_1q_2 + 0q_3 \end{pmatrix},$$

where the right-hand side can now be written as the juxtaposed, matrix-vector product form $-\underline{\mathbf{r}}\times\underline{\mathbf{q}}$, i.e.

$$\begin{pmatrix} 0q_1 + q_2r_3 - r_2q_3 \\ -r_3q_1 + 0q_2 + r_1q_3 \\ r_2q_1 - r_1q_2 + 0q_3 \end{pmatrix} = \begin{pmatrix} 0 & r_3 & -r_2 \\ -r_3 & 0 & r_1 \\ r_2 & -r_1 & 0 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = -\underline{\mathbf{r}} \times \underline{\mathbf{q}}\,.$$

This too is nothing more than a statement of the cross product property $\underline{\mathbf{q}} \times \underline{\mathbf{r}} = -\underline{\mathbf{r}} \times \underline{\mathbf{q}}$ (9.28)

Orthogonality

Moving away from $\underline{\mathbf{q}}\times$ and back to the complete, 4x4 representation matrix $\mathbf{Q_q}$ (10.1), this matrix has the property that the ith row is orthogonal to the jth row, when i and j are not equal. Likewise, the ith column is orthogonal to the jth column when i and j are not equal, the latter by virtue of the symmetry in $\mathbf{Q_q}$, disregarding the sign. This orthogonality is put to good use in URMT's treatment of rotations in three dimensions, Section (14), where columns two to four in $\mathbf{Q_q}$ form the three, zero eigenvectors in a URM5 representation - yes, that is a 5x5 representation and not 4x4, noting that URMT invariably adds an extra dimension when embedding an arbitrary vector (or a four-element quaternion).

Algebraically, the orthogonality of the rows and columns in $\mathbf{Q_q}$ means that the product of $\mathbf{Q_q}$ with its transpose, i.e. $\mathbf{Q_q^T Q_q}$, is a diagonal matrix. In this quaternionic case it also has the same, non-zero value for every element on the lead diagonal, namely the square of the quaternion magnitude, i.e. $|\mathbf{q}|^2$. This result can actually be deduced relatively simply using the earlier result $\mathbf{Q_q}^T = \mathbf{Q_q}^*$ (10.15) and $\mathbf{q}^*\mathbf{q} = |\mathbf{q}|^2$ (9.48), since

(10.31) $\mathbf{Q_q}^T\mathbf{Q_q} = \mathbf{Q_q}^*\mathbf{Q_q} \sim \mathbf{q}^*\mathbf{q} = |\mathbf{q}|^2 \sim \mathbf{Q}_{|\mathbf{q}|^2} = |\mathbf{q}|^2\mathbf{I}_4$.

However, as an algebraic workout, it is worth verifying this result longhand by using the definitions $\mathbf{Q_q}$ (10.10) and $\mathbf{Q_q}^T = \mathbf{Q}_0^T - \mathbf{Q}_{\underline{\mathbf{q}}}$ (10.14) to give

(10.32) $\mathbf{Q_q}^T\mathbf{Q_q} = \mathbf{Q}_0^2 - \mathbf{Q}_{\underline{\mathbf{q}}}^2$.

The $\mathbf{Q}_0^2$ term is simply obtained from (10.8) as

(10.33) $\mathbf{Q}_0^2 = q_0^2\mathbf{I}_4$.

and using the block definition of $\mathbf{Q}_{\underline{\mathbf{q}}}$ (10.9), then the matrix product $\mathbf{Q}_{\underline{\mathbf{q}}}^2$ is

(10.34) $\mathbf{Q}_{\underline{\mathbf{q}}}^2 = \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix}\begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix} = \begin{pmatrix} -|\underline{\mathbf{q}}|^2 & 0 \\ 0 & -\underline{\mathbf{q}}\underline{\mathbf{q}}^T + \underline{\mathbf{q}}\times\underline{\mathbf{q}}\times \end{pmatrix}$,

where the following relations have been used:

$\underline{\mathbf{q}}^T \underline{\mathbf{q}} = |\underline{\mathbf{q}}|^2$ by the inner product (9.40)

$\underline{\mathbf{q}} \times \underline{\mathbf{q}} = 0$ by the cross product (9.29)

$\underline{\mathbf{q}}^T (\underline{\mathbf{q}} \times) = 0$ by orthogonality - see below.

This last result is verified as follows, using the skew-symmetry of $\underline{\mathbf{q}}\times$, i.e. $(\underline{\mathbf{q}}\times)^T = -\underline{\mathbf{q}}\times$, and $\underline{\mathbf{q}} \times \underline{\mathbf{q}} = 0$:

(10.35) $\underline{\mathbf{q}}^T (\underline{\mathbf{q}}\times) = ((\underline{\mathbf{q}}\times)^T \underline{\mathbf{q}})^T = (-\underline{\mathbf{q}} \times \underline{\mathbf{q}})^T = 0$.

From the full definitions of $\underline{\mathbf{q}}$ (9.8) and $\underline{\mathbf{q}}\times$ (10.6), the terms $\underline{\mathbf{q}}\underline{\mathbf{q}}^T$ and $\underline{\mathbf{q}}\times\underline{\mathbf{q}}\times$ in the bottom right, 3x3 sub-matrix of $\mathbf{Q}_{\underline{\mathbf{q}}}^2$ are

(10.36) $$\underline{\mathbf{q}}\underline{\mathbf{q}}^T = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \begin{pmatrix} q_1 & q_2 & q_3 \end{pmatrix} = \begin{pmatrix} q_1^2 & q_1 q_2 & q_1 q_3 \\ q_2 q_1 & q_2^2 & q_2 q_3 \\ q_3 q_1 & q_3 q_2 & q_3^2 \end{pmatrix}$$

(10.37) $$\underline{\mathbf{q}}\times\underline{\mathbf{q}}\times = (\underline{\mathbf{q}}\times)^2 = \begin{pmatrix} -q_3^2 - q_2^2 & q_2 q_1 & q_3 q_1 \\ q_1 q_2 & -q_3^2 - q_1^2 & q_3 q_2 \\ q_1 q_3 & q_2 q_3 & -q_2^2 - q_1^2 \end{pmatrix}.$$

Upon evaluating the sum $-\underline{\mathbf{q}}\underline{\mathbf{q}}^T + \underline{\mathbf{q}}\times\underline{\mathbf{q}}\times$, all off-diagonal elements cancel to zero, leaving just a non-zero lead diagonal with every element equal to $-|\underline{\mathbf{q}}|^2$, as per the first element of $\mathbf{Q}_{\underline{\mathbf{q}}}^2$, i.e.

(10.38) $-\underline{\mathbf{q}}\underline{\mathbf{q}}^T + \underline{\mathbf{q}}\times\underline{\mathbf{q}}\times = -|\underline{\mathbf{q}}|^2 \mathbf{I}_3$.

The entire product $\mathbf{Q}_{\underline{\mathbf{q}}}^2$ (10.34) is therefore

(10.39) $\mathbf{Q}_{\underline{\mathbf{q}}}^2 = -|\underline{\mathbf{q}}|^2 \mathbf{I}_4$,

and so, using $\mathbf{Q}_0^2$ (10.33), then $\mathbf{Q}_{\mathbf{q}}^T \mathbf{Q}_{\mathbf{q}}$ (10.32) evaluates to

(10.40) $\mathbf{Q}_{\mathbf{q}}^T \mathbf{Q}_{\mathbf{q}} = \left(q_0^2 + |\underline{\mathbf{q}}|^2 \right) \mathbf{I}_4$,

and thus the lead diagonal comprises just the constant value $q_0^2 + |\underline{\mathbf{q}}|^2$. However, this is just the squared magnitude of the quaternion $\mathbf{q}$, i.e.

(10.41) $q_0^2 + |\underline{\mathbf{q}}|^2 = |\mathbf{q}|^2$, and so verifying (10.31). This result is expected because the transpose is also the quaternion conjugate, i.e. $\mathbf{Q}_{\mathbf{q}}^T = \mathbf{Q}_{\mathbf{q}}^*$ (10.15), hence the product $\mathbf{Q}_{\mathbf{q}}^T \mathbf{Q}_{\mathbf{q}} = \mathbf{Q}_{\mathbf{q}}^* \mathbf{Q}_{\mathbf{q}}$ will give a real-valued result, i.e. a quaternion matrix representation with a zero quaternion vector component ($\mathbf{Q}_{\underline{\mathbf{q}}} = 0$), and therefore a real-only, lead diagonal, actually equal to $|\mathbf{q}|^2$, as above.

This result also means that the transpose is identical to the inverse, barring a scale factor $|\mathbf{q}|^2$. More precisely, the inverse $\mathbf{Q}_{\mathbf{q}}^{-1}$ of $\mathbf{Q}_{\mathbf{q}}$ is thus

(10.42) $\mathbf{Q}_{\mathbf{q}}^{-1} = \dfrac{\mathbf{Q}_{\mathbf{q}}^T}{|\mathbf{q}|^2}$.

If the squared magnitude is unity then the transpose of $\mathbf{Q}_{\mathbf{q}}$ is identical to its inverse, i.e.

(10.43) $|\mathbf{q}| = 1 \Rightarrow \mathbf{Q}_{\mathbf{q}}^{-1} = \mathbf{Q}_{\mathbf{q}}^T$.

Such a matrix (with real elements), whose transpose is its inverse, is known as 'orthogonal'. A complex matrix, which is equal to its inverse when subject to both a transposition and a complex conjugation of all its elements, is known as 'unitary' - see Section (6-9).

Unit magnitude quaternions are used all the time in the representation of rotations in three dimensions since they preserve the length of the vector under transformation. For example, for a rotation of angle ρ about a unit vector axis $\hat{\mathbf{e}}$, the scalar and vector components are given by

(10.44) $q_0 = \cos(\rho/2)$, $\underline{\mathbf{q}} = \sin(\rho/2)\hat{\mathbf{e}}$ (11.3)

and hence the magnitude is unity, i.e.

(10.45) $q_0^2 + |\underline{\mathbf{q}}|^2 = \cos^2(\rho/2) + \sin^2(\rho/2) = 1$.

Lastly, the product $\mathbf{Q}_{\underline{\mathbf{q}}}^2$, evaluated earlier (10.39), rearranges to the following quadratic polynomial

(10.46) $\mathbf{Q}_{\underline{\mathbf{q}}}^2 + |\underline{\mathbf{q}}|^2 \mathbf{I}_4 = 0$,

which implies, by the Cayley Hamilton Theorem [5], that two of the eigenvalues of $\mathbf{Q}_{\underline{\mathbf{q}}}$ are thus

(10.47) $\lambda = \pm i|\underline{\mathbf{q}}|$.

In fact, this complex conjugate pair is repeated, and the four eigenvalues of $\mathbf{Q}_{\underline{\mathbf{q}}}$ are $\lambda = \pm i|\underline{\mathbf{q}}|$ repeated, which is in agreement with the eigenvalue solution given earlier (10.24).

(10-4) The Quaternion Product as a Matrix Product

The quaternion product $\mathbf{qr}$ was stated earlier (10.3) as the following matrix-vector product:

$$\mathbf{qr} = \mathbf{Q_q r} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{pmatrix} \text{ (10.3).}$$

However, this expression lacks some symmetry because the quaternion $\mathbf{q}$ is given by a matrix, whereas the quaternion $\mathbf{r}$ remains a vector. Although harmless, it would be nicer if both terms in the product were of the same type, i.e. a matrix. The downside of this is that the full matrix product is then overkill to evaluate the quaternion product since it calculates more terms than necessary. Nevertheless, it is useful to see how this works.

In an identical way to $\mathbf{q}$, represented by $\mathbf{Q_q}$, the quaternion $\mathbf{r}$ is represented by the matrix $\mathbf{Q_r}$, defined in full as

(10.48) $$\mathbf{Q_r} = \begin{pmatrix} r_0 & -r_1 & -r_2 & -r_3 \\ r_1 & r_0 & -r_3 & r_1 \\ r_2 & r_3 & r_0 & r_2 \\ r_3 & -r_1 & r_2 & r_0 \end{pmatrix},$$

and in block form as

(10.49) $$\mathbf{Q_r} = r_0 \mathbf{I}_4 + \begin{pmatrix} 0 & -\underline{\mathbf{r}}^T \\ \underline{\mathbf{r}} & \underline{\mathbf{r}}\times \end{pmatrix}.$$

The quaternion product $\mathbf{qr}$ then becomes the matrix product

(10.50) $$\mathbf{qr} \sim \mathbf{Q_q Q_r}$$

Expanding in block form, the full matrix product is thus

(10.51)

$$\mathbf{Q_q Q_r} = q_0 r_0 \mathbf{I}_4 + r_0 \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix} + q_0 \begin{pmatrix} 0 & -\underline{\mathbf{r}}^T \\ \underline{\mathbf{r}} & \underline{\mathbf{r}}\times \end{pmatrix} + \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix} \begin{pmatrix} 0 & -\underline{\mathbf{r}}^T \\ \underline{\mathbf{r}} & \underline{\mathbf{r}}\times \end{pmatrix}$$

The vector component, matrix product on the far right evaluates as

(10.52) $$\begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \underline{\mathbf{q}} & \underline{\mathbf{q}}\times \end{pmatrix} \begin{pmatrix} 0 & -\underline{\mathbf{r}}^T \\ \underline{\mathbf{r}} & \underline{\mathbf{r}}\times \end{pmatrix} = \begin{pmatrix} -\underline{\mathbf{q}}^T \mathbf{r} & -\underline{\mathbf{q}}^T \underline{\mathbf{r}}\times \\ \underline{\mathbf{q}}\times \mathbf{r} & -\underline{\mathbf{q}}\underline{\mathbf{r}}^T + \underline{\mathbf{q}}\times\underline{\mathbf{r}}\times \end{pmatrix}.$$

See (10.6) for the 3x3 matrix $\underline{\mathbf{q}}\times$, and replace $\underline{\mathbf{q}}$ with $\underline{\mathbf{r}}$ for $\underline{\mathbf{r}}\times$.

The type of each term in this product is briefly given as follows:

(10.53)

- $\underline{\mathbf{q}}^T\underline{\mathbf{r}} = \underline{\mathbf{q}}\cdot\underline{\mathbf{r}}$: the inner (dot) product between $\underline{\mathbf{q}}$ and $\underline{\mathbf{r}}$ giving a scalar
- $\underline{\mathbf{q}}^T\underline{\mathbf{r}}\times$: the product of row vector $\underline{\mathbf{q}}^T$ and matrix $\underline{\mathbf{r}}\times$ giving a row vector
- $\underline{\mathbf{q}}\underline{\mathbf{r}}^T$: an outer product giving a 3x3 matrix
- $\underline{\mathbf{q}}\times\underline{\mathbf{r}}\times$: a matrix product of two, 3x3 matrices $\underline{\mathbf{q}}\times$ and $\underline{\mathbf{r}}\times$ giving a 3x3 matrix

It is not actually necessary to calculate the full matrix product $\mathbf{Q_q Q_r}$ because the resulting matrix result itself should be a matrix representation of the resulting quaternion product. Thus, by comparing the result with the quaternion matrix representation (10.7), the top left element gives the scalar result, and the bottom left vector gives the vector result. Collecting only these two, left column terms therefore gives

(10.54) $$\mathbf{Q_q Q_r} = \begin{pmatrix} q_0 r_0 - \underline{\mathbf{q}}^T\underline{\mathbf{r}} & - \\ r_0\underline{\mathbf{q}} + q_0\underline{\mathbf{r}} + \underline{\mathbf{q}}\times\underline{\mathbf{r}} & - \end{pmatrix}.$$

Comparing this with the quaternion product $\mathbf{qr}$ (9.30), using $\underline{\mathbf{q}}^T\underline{\mathbf{r}} = \underline{\mathbf{q}}\cdot\underline{\mathbf{r}} = \underline{\mathbf{r}}\cdot\underline{\mathbf{q}} = \underline{\mathbf{r}}^T\underline{\mathbf{q}}$ (9.25), confirms the above assumption that the left-hand column gives the quaternion product.

Lastly, one good reason to use an all-matrix representation, instead of using the quaternion multiplication equation, is that no knowledge of quaternion multiplication rules are required, only the standard rules of matrix multiplication.

11 Quaternions and Single Axis Rotations

(11-1) A quaternion as a rotation

Quaternions are used to represent rotations in three dimensions, just like complex numbers can be used to represent planar rotations in one dimension, i.e. rotations about a single axis. This section is an overview of how this representation is achieved, starting with planar rotations and then moving on to rotations about three different axes. Lastly, and most important to URMT's usage of quaternions, this section examines angular rates as background to the two sections that follow on the subject of particle spin and rigid body motion.

(11.1) **Notation**. The following three quaternions are used throughout this section:

quaternion $\mathbf{q}$: scalar q_0, vector $\underline{\mathbf{q}}$, i.e. $\mathbf{q} = (q_0, \underline{\mathbf{q}})^T$
quaternion $\mathbf{r}$: scalar r_0, vector $\underline{\mathbf{r}}$, i.e. $\mathbf{r} = \mathbf{r}(r_0, \underline{\mathbf{r}})^T$
quaternion $\mathbf{u}$: scalar u_0, vector $\underline{\mathbf{u}}$, i.e. $\mathbf{u} = (u_0, \underline{\mathbf{u}})^T$

Using the following definitions:

(11.2)
$\hat{\mathbf{e}}$: a unit vector aligned along the axis of rotation

ρ: the angle of rotation about $\hat{\mathbf{e}}$, positive clockwise when looking along the vector outward from the origin, or anticlockwise when looking toward the origin.

then the quaternion $\mathbf{q}$ is a representation of this rotation when q_0 and $\underline{\mathbf{q}}$ are given by

(11.3) $q_0 = \cos(\rho/2)$, $\underline{\mathbf{q}} = \sin(\rho/2)\hat{\mathbf{e}}$

With these definitions, the quaternion representing the rotation, symbol $\mathbf{q}_\rho$, is thus

(11.4) $\mathbf{q}_\rho = \cos(\rho/2) + \sin(\rho/2)\hat{\mathbf{e}}$,

alternatively written in block vector form as

(11.5) $\mathbf{q}_\rho = \begin{pmatrix} \cos(\rho/2) \\ \sin(\rho/2)\hat{\mathbf{e}} \end{pmatrix}$.

(11.6)
The half angle $\rho/2$ is intentional, and is usually described (very loosely) in texts along the lines of "*a single rotation in the real world requires two rotations in the quaternion world to get back to the starting point*". Algebraically, when using quaternions to rotate a vector, this '*two-rotations*' aspect manifests itself in the transformation (rotation) process as two multiplications, one by the quaternion and another by its conjugate, which causes the doubling of the half angle $\rho/2$ to give ρ; see

$\mathbf{a}' = \mathbf{q}_\rho \mathbf{a} \mathbf{q}_\rho^*$ (11.12) below. Furthermore, the factor of two also arises naturally in URMT's eigenvector relation $\mathbf{A}_+\mathbf{X}_- = 2C\mathbf{X}_0$ (14.14) – see Section (14-6) for its application to quaternion rates and rigid body rotations. Lastly on this half-angle subject, see [8] for a topological argument.

The specifics of exactly how the rotation representation is achieved by $\mathbf{q}_\rho$ is now given in more detail by studying how the components of a vector transform when rotated.

(11-2) Transformation of Vector Components under a Rotation

If $\underline{\mathbf{a}}$ denotes an arbitrary, Cartesian vector,

(11.9) $\underline{\mathbf{a}} = xi + yj + zk$,

and written as a quaternion $\mathbf{a}$, with a zero, real scalar part, $a_0 = 0$, and non-zero vector part $\underline{\mathbf{a}}$, i.e.

(11.10) $\mathbf{a} = \underline{\mathbf{a}} = \begin{pmatrix} 0 \\ \underline{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix}$,

then the following product, where $\mathbf{q}_\rho^*$ is the quaternion conjugate of $\mathbf{q}_\rho$, i.e.

(11.11) $\mathbf{q}_\rho^* = \begin{pmatrix} \cos(\rho/2) \\ -\sin(\rho/2)\hat{\mathbf{e}} \end{pmatrix}$,

gives the rotated vector $\underline{\mathbf{a}}'$ as the quaternion $\mathbf{a}'$:

(11.12) $\mathbf{a}' = \mathbf{q}_\rho \mathbf{a} \mathbf{q}_\rho^* = \begin{pmatrix} 0 \\ \underline{\mathbf{a}}' \end{pmatrix}$.

This operation rotates the vector $\underline{\mathbf{a}}$ by a full angle ρ about the axis $\hat{\mathbf{e}}$ to a new vector $\underline{\mathbf{a}}'$. Because the vector is rotated, not the axes frame, this is known as an 'active' rotation; 'passive' rotations are considered soon – see Appendix (H2) for definitions. It is because the multiplication effectively involves $\mathbf{q}_\rho$ twice - the second time by its conjugate $\mathbf{q}_\rho^*$ - that the half-angle $\rho/2$ becomes the full angle ρ, as will be algebraically demonstrated shortly.

The action of (11.12) can be seen in detail by considering the rotation of a vector, lying in the x-y plane, about the z axis. The rotation axis is then simply the unit vector

(11.13) $\hat{\mathbf{e}} = k = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Substituting for $\hat{\mathbf{e}}$ into (11.5) gives the quaternion $\mathbf{q}_\rho$ and its conjugate $\mathbf{q}_\rho^*$ as

$$(11.14)\quad \mathbf{q}_\rho = \begin{pmatrix} \cos(\rho/2) \\ 0 \\ 0 \\ \sin(\rho/2) \end{pmatrix},\ \mathbf{q}_\rho^* = \begin{pmatrix} \cos(\rho/2) \\ 0 \\ 0 \\ -\sin(\rho/2) \end{pmatrix}.$$

To make this example as simple as possible, the initial vector $\underline{\mathbf{a}}$ shall lie along the x axis, with no y or z component, i.e.

$$(11.15)\quad \underline{\mathbf{a}} = xi = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}.$$

For a positive rotation of angle ρ, using $\mathbf{q}$ and its conjugate $\mathbf{q}^*$ (11.14), but written longhand as

$$(11.16)\quad \mathbf{q}_\rho = \cos(\rho/2) + \sin(\rho/2)k$$

$$(11.17)\quad \mathbf{q}_\rho^* = \cos(\rho/2) - \sin(\rho/2)k,$$

then the rotated vector $\underline{\mathbf{a}}'$ (as a quaternion $\mathbf{a}'$) is calculated from the quaternion product (11.12), expanded in full as

$$(11.18)\quad \mathbf{a}' = \left(\cos(\rho/2) + \sin(\rho/2)k\right)\left(xi\right)\left(\cos(\rho/2) - \sin(\rho/2)k\right).$$

Multiplying out the last two bracketed terms, and using the quaternion product rule $ik = -j$ (9.1d), gives

$$(11.19)\quad \mathbf{a}' = \left(\cos(\rho/2) + \sin(\rho/2)k\right)\left(x\cos(\rho/2)i + x\sin(\rho/2)j\right),$$

Further expanding this using $ki = j$ and $kj = -i$ (9.1c) then

$$(11.20)\quad \mathbf{a}' = x\left(\cos^2(\rho/2) - \sin^2(\rho/2)\right)i + 2x\cos(\rho/2)\sin(\rho/2)j.$$

Lastly, using the two trigonometric identities

$$(11.21)\quad \cos 2\theta = \cos^2\theta - \sin^2\theta,\ \sin 2\theta = 2\cos\theta\sin\theta,$$

then $\mathbf{a}'$ simplifies to just

$$(11.22)\quad \mathbf{a}' = x\cos(\rho)i + x\sin(\rho)j.$$

Thus, the resulting rotated vector is given in quaternion form as

(11.23) $\mathbf{a}' = \begin{pmatrix} 0 \\ x\cos(\rho) \\ x\sin(\rho) \\ 0 \end{pmatrix}$, quaternion

or in standard vector form as

(11.24) $\underline{\mathbf{a}}' = \begin{pmatrix} x\cos(\rho) \\ x\sin(\rho) \\ 0 \end{pmatrix}$, vector.

The quaternion $\mathbf{a}'$ has no real scalar component, i.e. $a_0' = 0$, and comprises the pure vector component $\underline{\mathbf{a}}'$, which is the transformed vector as would be obtained using standard vector algebra to rotate the vector. Quaternion $\mathbf{a}'$ is thus written in block vector form as

(11.25) $\mathbf{a}' = \begin{pmatrix} 0 \\ \underline{\mathbf{a}}' \end{pmatrix}$.

Doing the same for a vector lying along the y axis, with no initial x or z component, i.e.

(11.26) $\underline{\mathbf{a}} = yj = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix}$,

then the transformed vector $\underline{\mathbf{a}}'$ is obtained in exactly the same way by calculating the equivalent quaternion transformation (11.12)

(11.27) $\mathbf{a}' = \left(\cos(\rho/2) + \sin(\rho/2)k\right)\left(yj\right)\left(\cos(\rho/2) - \sin(\rho/2)k\right)$,

which results in the transformed vector

(11.28) $\underline{\mathbf{a}}' = \begin{pmatrix} -y\sin(\rho) \\ y\cos(\rho) \\ 0 \end{pmatrix}$.

Since the two vector forms of $\underline{\mathbf{a}}'$, (11.24) and (11.28), can be added together (see the following note), the combined result for the transformation of a vector $\underline{\mathbf{a}}$ lying in the x-y plane is

(11.29) $\underline{\mathbf{a}}' = \begin{pmatrix} x' \\ y' \\ 0 \end{pmatrix} = \begin{pmatrix} x\cos(\rho) \\ x\sin(\rho) \\ 0 \end{pmatrix} + \begin{pmatrix} -y\sin(\rho) \\ y\cos(\rho) \\ 0 \end{pmatrix}$.

Note that it was stated earlier that quaternions cannot be meaningfully added when representing angles. However, this is precisely what has just been done above. Nevertheless, this is permissible when the two separate rotations are about the same axis, as is the case here; see also sub-section (11-4) below.

Finally then, the components x, y of the planar vector $\underline{\mathbf{a}}$ transform to components x', y' of the rotated vector $\underline{\mathbf{a}}'$ (11.29), when $\underline{\mathbf{a}}$ is rotated about the z axis, as follows:

(11.30)

$$x' = x\cos\rho - y\sin\rho$$

$$y' = x\sin\rho + y\cos\rho$$

and since any z component of a 3D vector remains unchanged by a rotation about the z axis then, of course, its transformed component z' is unchanged, i.e.

(11.31) $z' = z$.

All three components x, y, z thus transform as follows, now written in matrix form as

(11.32) $$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\rho & -\sin\rho & 0 \\ \sin\rho & \cos\rho & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

This result can also be obtained in a single step by calculating the full quaternion product

(11.33) $$\mathbf{a}' = \left(\cos(\rho/2) + \sin(\rho/2)k\right)\left(xi + yj + zk\right)\left(\cos(\rho/2) - \sin(\rho/2)k\right).$$

The matrix in (11.32) is, unsurprisingly, the same as a rotation matrix, e.g. (H4), but with $\psi \to -\psi$ for an active rotation of a vector about the z axis.

For a passive rotation (H2), all that is required is the inversion of the sign of the angle, i.e. $\rho \to -\rho$ in the quaternion $\mathbf{q}_\rho$ (11.5), which has the effect of reversing the sign of the vector term $\underline{\mathbf{q}} = \sin(\rho/2)\hat{\mathbf{e}}$ since, for any angle θ, $\sin(-\theta) = -\sin(\theta)$ and $\cos(-\theta) = \cos(\theta)$. This then is just the same as taking the quaternion conjugate. In other words, an active rotation represented by quaternion $\mathbf{q}_\rho$ is the same as a passive rotation with a sign reversal, i.e. $\mathbf{q}_{-\rho}$ $\left(= \mathbf{q}^*\right)$. In practice, in all quaternion representations and direction cosine matrices (for a single axis rotation), e.g. (H5), this means just replacing the $\sin(\theta)$ terms with $-\sin(\theta)$ wherever they occur. Note that 'θ' here is an arbitrary angle, not necessarily the pitch angle.

The above is a simple planar rotation example and, as such, the same result could be obtained using complex numbers. However, quaternions were explicitly developed for their 3D rotation properties, and it is only in the 3D case do they come into their own. Full 3D rotations are discussed in Section (14) but, beforehand, the intermediate step of compound rotations is now detailed.

(11-3) Compound Rotations

Following on from the single rotation, represented by $\mathbf{q}_\rho$ (11.5), if a second quaternion $\mathbf{r}_\sigma$ represents a rotation of angle σ about an axis $\underline{\mathbf{r}}$, then the compound rotation of $\mathbf{q}_\rho$ followed by $\mathbf{r}_\sigma$ is equivalent to a single rotation through an angle ϑ about an axis $\underline{\mathbf{u}}$, represented by quaternion $\mathbf{u}_\vartheta$, where $\mathbf{u}_\vartheta$ is given by the quaternion product

(11.34) $\mathbf{u}_\vartheta = \mathbf{r}_\sigma \mathbf{q}_\rho$.

This can be seen by rotating the vector $\underline{\mathbf{a}}$ (represented by quaternion $\mathbf{a}$ (11.10)) in two stages. The first rotation of $\underline{\mathbf{a}}$ through angle ρ about axis $\underline{\mathbf{q}}$, represented by quaternion $\mathbf{q}_\rho$, is as before (11.12), transforming quaternion $\mathbf{a}$ to $\mathbf{a}'$, i.e.

$$\mathbf{a}' = \mathbf{q}_\rho \mathbf{a} \mathbf{q}_\rho^* \quad (11.12)$$

In a similar way, vector $\underline{\mathbf{a}}'$ (quaternion $\mathbf{a}'$) is then rotated through angle σ about axis $\underline{\mathbf{r}}$ to give vector $\underline{\mathbf{a}}''$ (quaternion $\mathbf{a}''$), i.e.

(11.35) $\mathbf{a}'' = \mathbf{r}_\sigma \mathbf{a}' \mathbf{r}_\sigma^*$.

Substituting for $\mathbf{a}'$ from (11.12) into this expression gives

(11.36) $\mathbf{a}'' = \mathbf{r}_\sigma \mathbf{q}_\rho \mathbf{a} \mathbf{q}_\rho^* \mathbf{r}_\sigma^*$,

and by the associativity of quaternion algebra (9.42) this can be grouped as

(11.37) $\mathbf{a}'' = (\mathbf{r}_\sigma \mathbf{q}_\rho)\mathbf{a}(\mathbf{q}_\rho^* \mathbf{r}_\sigma^*)$.

Using the compound conjugation rule $(\mathbf{q}_\rho^* \mathbf{r}_\sigma^*) = (\mathbf{r}_\sigma \mathbf{q}_\rho)^*$ (9.54), then $\mathbf{a}''$ becomes

(11.38) $\mathbf{a}'' = (\mathbf{r}_\sigma \mathbf{q}_\rho)\mathbf{a}(\mathbf{r}_\sigma \mathbf{q}_\rho)^*$.

Lastly, using the definition (11.34) of quaternion $\mathbf{u}_\vartheta$, then $\mathbf{a}''$ is written as the product

(11.40) $\mathbf{a}'' = \mathbf{u}_\vartheta \mathbf{a} \mathbf{u}_\vartheta^*$,

and thus the compound rotation is equivalent to a single rotation represented by quaternion $\mathbf{u}_\vartheta$.

Notes

The order of the multiplication is important since consecutive rotations do not generally commute, except when about the same, single axis, i.e.

$$\mathbf{r}_\sigma \mathbf{q}_\rho \neq \mathbf{q}_\rho \mathbf{r}_\sigma .$$

See Appendix (H) for more details on rotation ordering.

The resultant rotation axis $\underline{\mathbf{u}}$ and angle ϑ can be extracted from the quaternion product, both of which will be functions of the angles ρ and σ, and the individual axis vectors $\underline{\mathbf{q}}$ and $\underline{\mathbf{r}}$.

This compound rotation is a multiplicative, binary operation, and not additive, i.e. the addition of the two quaternions, e.g. $\mathbf{r}_\sigma + \mathbf{q}_\rho$, does not give a physically meaningful result in terms of rotations, even though addition is a perfectly valid quaternion operation.

(11-4) Consecutive Rotations about the Same Axis

The above compound rotation (11.34) is a general case, with the axis vectors $\underline{\mathbf{q}}$ and $\underline{\mathbf{r}}$ completely arbitrary. Before moving on to a full, 3D, compound transformation in the next section, i.e. three consecutive rotations, each about a different axis, it is worth simplifying the above generalisation for a brief look at consecutive rotations about the same axis, i.e. when $\underline{\mathbf{q}}$ and $\underline{\mathbf{r}}$, and subsequently $\underline{\mathbf{u}}$, are the same unit vector. In this case, performing a rotation through angle ρ, followed by a rotation through angle σ about the same axis, is equivalent to the single rotation ϑ given by the algebraic sum of the two angles, i.e.

(11.41) $\vartheta = \sigma + \rho$,

and since rotations about the same axis commute then

(11.42) $\vartheta = \sigma + \rho = \rho + \sigma$,

and thus

(11.43) $\mathbf{u}_{\rho+\sigma} = \mathbf{r}_\sigma \mathbf{q}_\rho = \mathbf{q}_\rho \mathbf{r}_\sigma = \mathbf{u}_{\sigma+\rho}$.

This expression has exactly the same form as the exponential relation, $e^\sigma e^\rho = e^{\sigma+\rho}$, and is a reason why a complex-number representation can be used for planar (single axis) rotations, which use the complex representation $e^{i\rho}$ and $e^{i\sigma}$ for the same rotations.

A simple example is the case of performing the same angular rotation twice in succession, i.e. the angle σ is the same as ρ, and the quaternions $\mathbf{q}_\rho$ and $\mathbf{r}_\sigma$ are therefore also the same since both the angle and vector axis are now equal, i.e.

(11.44) $\mathbf{q}_\rho = \mathbf{r}_\sigma,\ \rho = \sigma$.

The resultant quaternion representing the repeated rotation is thus

(11.45) $\mathbf{u}_{2\rho} = \mathbf{q}_\rho^2$

Following from this it can be inferred that a finite rotation through angle ρ can also be composed of n consecutive rotations through the smaller angle ρ / n. Thus, the quaternion representation for a rotation through an angle ρ becomes the product of the quaternion representation for an angle ρ / n, repeated n times, i.e.

(11.46) $\mathbf{u}_{\rho} = \left(\mathbf{q}_{\rho / n}\right)^{n}$.

Once again, this is no different to that of a complex number in the single axis rotation case, i.e.

(11.47) $z(\rho) = \left(e^{i\rho / n}\right)^{n} = e^{i\rho}$.

In brief then, single axis rotations are commutative, as are the equivalent complex number representations - complex numbers commute. However, quaternion multiplication is not, in general, commutative, and neither do successive rotations about different axes commute. Thus, any algebra that represents rotations must have this non-commutative property. Matrix multiplication has this non-commutative property, and is therefore one reason why matrices are used to represent rotations, notably as direction cosine matrices (DCMs), as detailed in Appendix (H), e.g. (H11). Perhaps then it is no surprise that matrices can also represent quaternions, and the matrix representation of a quaternion is actually of prime interest in URMT because URMT is heavily focused on matrices, their eigenvalues, eigenvectors and the invariants that arise from their inner product relations, Appendix (F). These aspects are discussed at length in application to particle spin and general rigid body dynamics, Sections (13) and (14) respectively. However, for now, the study moves on to the representation of three dimensional rotations using quaternions.

12 Quaternions and Three Axis Rotations

(12-1) Pre-requisite

Before starting this section, readers are advised to peruse the angular dynamics primer, Appendix (H), particularly with regard to the nomenclature and conventions used in rotations.

(12-2) Compound Transformations and 3D Rotations

From the previous section, the quaternion $\mathbf{q}_\rho$ used to represent a rotation of angle ρ about a single axis $\hat{\mathbf{e}}$ is given by

$$\mathbf{q}_\rho = \cos(\rho/2) + \sin(\rho/2)\hat{\mathbf{e}} \text{ (11.4)},$$

alternatively written in block vector form as

$$\mathbf{q}_\rho = \begin{pmatrix} \cos(\rho/2) \\ \sin(\rho/2)\hat{\mathbf{e}} \end{pmatrix} \text{ (11.5)}.$$

Using the aerospace convention given in Appendix (H) then, for a passive rotation of the axes, the compound rotation to go from space to body axes comprises:

1) a yaw through angle ψ about the z axis (unit vector k), represented by quaternion $\mathbf{q}_\psi$

(12.1) $\mathbf{q}_\psi = \cos(\psi/2) + \sin(\psi/2)k$, yaw,

2) followed by a pitch through angle θ about the new y axis (unit vector j), represented by quaternion $\mathbf{q}_\theta$

(12.2) $\mathbf{q}_\theta = \cos(\theta/2) + \sin(\theta/2)j$, pitch,

3) followed by a roll through angle ϕ about the new x axis (unit vector i), represented by quaternion $\mathbf{q}_\phi$

(12.3) $\mathbf{q}_\phi = \cos(\phi/2) + \sin(\phi/2)i$, roll.

Multiplying these three quaternions in the appropriate order gives the quaternion representation '$\mathbf{q}_{\psi\theta\phi}$' of the passive, compound transformation as

(12.4) $\mathbf{q}_{\psi\theta\phi} = \mathbf{q}_\psi \mathbf{q}_\theta \mathbf{q}_\phi$.

Note that the quaternion multiplication ordering for this passive transformation is reversed from that of the active, compound transformation (11.34).

Writing the quaternion $\mathbf{q}_{\psi\theta\phi}$ in the usual way, as a four-element column vector comprising the four components (q_0, q_1, q_2, q_3), i.e.

(12.5) $\mathbf{q} = \mathbf{q}_{\psi\theta\phi} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$,

then the quaternion product, in terms of the components, is

(12.6)

$$q_0 = \cos(\psi/2)\cos(\theta/2)\cos(\phi/2) + \sin(\psi/2)\sin(\theta/2)\sin(\phi/2)$$

$$q_1 = \cos(\psi/2)\cos(\theta/2)\sin(\phi/2) - \sin(\psi/2)\sin(\theta/2)\cos(\phi/2)$$

$$q_2 = \cos(\psi/2)\sin(\theta/2)\cos(\phi/2) + \sin(\psi/2)\cos(\theta/2)\sin(\phi/2)$$

$$q_3 = \sin(\psi/2)\cos(\theta/2)\cos(\phi/2) - \cos(\psi/2)\sin(\theta/2)\sin(\phi/2).$$

The Euler angles (ϕ, θ, ψ) can be obtained from the quaternion components by

(12.7)

$$\tan(\phi) = \frac{2(q_2 q_3 + q_0 q_1)}{q_0^2 - q_1^2 - q_2^2 + q_3^2}$$

$$\sin(\theta) = 2(q_0 q_2 - q_1 q_3)$$

$$\tan(\psi) = \frac{2(q_1 q_2 + q_0 q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2}.$$

Using the unity-magnitude relation

(12.8) $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$,

then these equations can also be written as

(12.9)

$$\tan(\phi) = \frac{(q_2 q_3 + q_0 q_1)}{1/2 - q_1^2 - q_2^2}.$$

$$\sin(\theta) = 2(q_0 q_2 - q_1 q_3)$$

$$\tan(\psi) = \frac{(q_1 q_2 + q_0 q_3)}{1/2 - q_2^2 - q_3^2}$$

Note that these equations, (12.7) and (12.9), are not immune to the 90 deg pitch angle singularity that occurs when the denominator in the roll and yaw expressions is zero - see also (H29) and (H54) in Appendix (H). However, also note that, whilst quaternions resolve the yaw-rate issue discussed in Appendix (H), they cannot resolve this singularity issue when converting from quaternions to Eulers at exactly 90 deg pitch.

Before proceeding, a quick recap on notation as first given Section (9). The quaternion $\mathbf{q}$ comprises a scalar part q_0 and a vector part $\underline{\mathbf{q}}$, the combination written as

$$\mathbf{q} = q_0 + \underline{\mathbf{q}} \quad (9.7).$$

The vector part is often written in the familiar, Cartesian 'i,j,k' vector form as

$$\underline{\mathbf{q}} = q_1 i + q_2 j + q_3 k \quad (9.6),$$

but as used herein it is generally written as a three-element column vector, i.e.

$$\underline{\mathbf{q}} = \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad (9.8),$$

Using this notation, the quaternion $\mathbf{q}$ is then written in the block vector form as

$$\mathbf{q} = \begin{pmatrix} q_0 \\ \underline{\mathbf{q}} \end{pmatrix} \quad (9.10).$$

For rotations, the magnitude $|\mathbf{q}|$ of the quaternion $\mathbf{q}$ is always unity, which ensures that it preserves the length of vectors under transformation (rotation). However, for general quaternions, the magnitude is arbitrary and, as such, $|\mathbf{q}|$ will usually be used in place of unity throughout, although the reader can safely assume that $|\mathbf{q}| = 1$ in the context of rotations, unless stated otherwise.

(12-3) How the components of an arbitrary vector change under rotation of the axes.

If $\underline{\mathbf{a}}$ denotes an arbitrary, Cartesian vector,

$$\underline{\mathbf{a}} = xi + yj + zk \quad (11.9)$$

and written as a quaternion $\mathbf{a}$, with a zero, real scalar part, $a_0 = 0$, and vector part $\underline{\mathbf{a}}$, i.e.

$$\mathbf{a} = \underline{\mathbf{a}} = \begin{pmatrix} 0 \\ \underline{\mathbf{a}} \end{pmatrix} = \begin{pmatrix} 0 \\ x \\ y \\ z \end{pmatrix} \quad (11.10),$$

then the following product, where $(\mathbf{q}_{\psi\theta\phi})^*$ is the quaternion conjugate of $\mathbf{q}_{\psi\theta\phi}$ (12.5), gives the components of the vector $\underline{\mathbf{a}}'$ due to the compound rotation (ϕ, θ, ψ) of the axes (passive rotation):

$$(12.11) \quad \mathbf{a}' = (\mathbf{q}_{\psi\theta\phi})^* \mathbf{a} \mathbf{q}_{\psi\theta\phi} = \begin{pmatrix} 0 \\ \underline{\mathbf{a}}' \end{pmatrix}.$$

(12-4) Quaternion Rates

From the theory of quaternions [15], if $\mathbf{q}$ denotes a four-element quaternion vector, and $\dot{\mathbf{q}}$ its rate, i.e.

(12.13) $$\mathbf{q} = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}, \ \dot{\mathbf{q}} = d\mathbf{q}/dt = \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix},$$

then the quaternion rate vector $\dot{\mathbf{q}}$ can be expressed in terms of $\mathbf{q}$ using the following 4x4 body rate matrix $\mathbf{W}$, whose elements comprise solely of the body rates (p, q, r) (H43), as follows:

(12.14) $$\mathbf{W} = \begin{pmatrix} 0 & -p & -q & -r \\ p & 0 & r & -q \\ q & -r & 0 & p \\ r & q & -p & 0 \end{pmatrix}$$, quaternion rate matrix.

p : roll rate about the body fixed x axis
q : pitch rate about the body fixed y axis
r : yaw rate about the body fixed z axis.

The corresponding quaternion rate equation is given by

(12.15) $$\dot{\mathbf{q}} = \frac{\mathbf{W}}{2}\mathbf{q}.$$

The axes rotations are actually done in the yaw, pitch, roll order (yaw first about z, roll last about x) going from space axes to body-fixed axes. Body fixed axes rotate with the body, whereas space axes are inertial axes that do not rotate, and may only have a constant linear velocity, i.e. they do not accelerate. It is usual to set the origin coincident with the space axis, or at least an origin with which the space axes are at rest so that the space axes have no linear motion, i.e. zero velocity. For rotational problems, as detailed herein, both the space and body axes have coincident origins for all time, so there is no relative, linear motion between them.

This (12.15) is how the quaternion rates are often calculated in texts on the practical application of quaternions to angular dynamics, where the usage of symbols p, q and r in matrix $\mathbf{W}$ is popular, notably in aerospace.

Note that the body rates p, q and r are not the same as the URMT dynamical variables, P, Q and R (A1b), although the notation is not unrelated as it was realised early in the development of URM3 [1] that the matrix and its velocity (angular velocity) properties were similar - see Section (6-4) and (6.44).

All three body rates body rates p, q and r are better written (for further mathematical purposes) as the vector components ω_1, ω_2 and ω_3 of a body rate vector $\boldsymbol{\omega}$

(12.16)

$p \equiv \omega_1$, $q \equiv \omega_2$, $r \equiv \omega_3$ (6.44)

$$\boldsymbol{\omega} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \equiv \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \text{ body rate vector.}$$

Usage of vector components ω_1, ω_2 and ω_3 will shortly be seen to demystify $\mathbf{W}$, which is now equivalently written as

$$(12.17)\ \mathbf{W} = \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -\boldsymbol{\omega}\times \end{pmatrix} \text{ (12.28).}$$

Writing the rate equation (12.15) using this form of $\mathbf{W}$ gives

$$(12.18)\ \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}.$$

By expanding this transformation out in full as four linear equations, it is then easily verified that it can also be written in the following, reversed form, i.e. where the matrix now contains the quaternion components, and the body rates form the vector, i.e.

$$(12.19)\ \begin{pmatrix} \dot{q}_0 \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} 0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

By reference back to Section (10), when written this way the matrix is seen to be a true quaternion representation matrix, denoted by $\mathbf{Q_q}$ (10.1), reproduced below.

$$\mathbf{Q_q} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \text{ (10.1)}$$

Such a matrix is nicer written in the more concise, block matrix form in terms of the quaternion scalar q_0 and vector $\underline{\mathbf{q}}$ components as follows, reproduced from Section (10):

$$(12.20)\ \mathbf{Q_q} = q_0 \mathbf{I}_4 + \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \mathbf{q} & \underline{\mathbf{q}}\times \end{pmatrix} \text{ (10.5),}$$

where the 3x3, bottom right sub-matrix ' $\underline{\mathbf{q}}\times$' is the cross product representation matrix (10.6).

There are two, 4x4 matrices on the right of (12.20) defined as $\mathbf{Q}_0$ and $\mathbf{Q}_{\underline{\mathbf{q}}}$

$$\mathbf{Q}_0 = q_0 \mathbf{I}_4 \text{ (10.8)}, \quad \mathbf{Q}_{\underline{\mathbf{q}}} = \begin{pmatrix} 0 & -\underline{\mathbf{q}}^T \\ \mathbf{q} & \underline{\mathbf{q}}\times \end{pmatrix} \text{ (10.9)}.$$

Together they sum to a quaternion representation of the complete quaternion, scalar q_0 and vector component $\underline{\mathbf{q}}$:

$$\mathbf{Q}_{\mathbf{q}} = \mathbf{Q}_0 + \mathbf{Q}_{\underline{\mathbf{q}}} \text{ (10.10)}.$$

By defining the body rate quaternion $\mathbf{q}_{\boldsymbol{\omega}}$ in terms of the body rate vector $\boldsymbol{\omega}$, with a zero scalar part, i.e.

(12.21) $\mathbf{q}_{\boldsymbol{\omega}} = \begin{pmatrix} 0 \\ \boldsymbol{\omega} \end{pmatrix}, \; q_0 = 0$,

then the rate equation (12.15) is now written as the quaternion matrix/vector product

(12.22) $\dot{\mathbf{q}} = \dfrac{\mathbf{Q}_{\mathbf{q}}}{2}\mathbf{q}_{\boldsymbol{\omega}}$.

Using $\mathbf{Q}_{\mathbf{q}}$ (12.20), this rate equation is expanded in a vector form as

(12.23) $\dot{\mathbf{q}} = q_0\boldsymbol{\omega} - \underline{\mathbf{q}}^T\boldsymbol{\omega} + \mathbf{q}\times\boldsymbol{\omega}$,

where $\underline{\mathbf{q}}^T\boldsymbol{\omega}$ is the inner vector product, i.e. $\underline{\mathbf{q}}^T\boldsymbol{\omega} = \underline{\mathbf{q}}\cdot\boldsymbol{\omega}$, and forms the scalar part of the quaternion rate $\dot{\mathbf{q}}$, also written as

(12.24) $\dot{\mathbf{q}} = \begin{pmatrix} \dot{q}_0 \\ \dot{\underline{\mathbf{q}}} \end{pmatrix} = \begin{pmatrix} -\underline{\mathbf{q}}\cdot\boldsymbol{\omega} \\ q_0\boldsymbol{\omega} + \mathbf{q}\times\boldsymbol{\omega} \end{pmatrix}$.

Returning to equation (12.19), this shows that the quaternion represented by $\mathbf{Q}_{\mathbf{q}}$ performs a linear transformation on the body rate $\boldsymbol{\omega}$ converting it to its physically equivalent quaternion rate. The point here being that the initial and final quaternion have the same physical units of frequency, with the quaternion representation matrix being dimensionless. The physical units of the quantities given so far are as follows:

(12.25)

$units(\boldsymbol{\omega}) = T^{-1}$, frequency

$units(\mathbf{W}) = T^{-1}$, frequency

$unit(\dot{\mathbf{q}}) = T^{-1}$, frequency.

$units(\mathbf{Q_q}) =$ none, dimensionless

$unit(\mathbf{q_\omega}) =$ none, dimensionless.

In contrast to (12.19), the first form of the rate equation $\dot{\mathbf{q}} = (\mathbf{W}/2)\mathbf{q}$ (12.15), given in terms of the body rate matrix **W**, physically transforms the dimensionless quaternion to its equivalent rate. Such an action is equivalent to a derivative operation, and thus the matrix $\mathbf{W}/2$ performs the equivalent action to a derivative operator. This is now the realm of URMT because URMT also has a matrix $\mathbf{A}_+$ that is equivalent to a derivative operator (6.2). This calculus nature is most important to URMT when discussing rigid body rotations in Section (14).

By comparing both matrix forms **W** (12.17) and $\mathbf{Q_q}$ (10.1), it is clear from the vector indices that they are very similar barring the fact that **W** has an all-zero lead-diagonal, whereas $\mathbf{Q_q}$ has a lead diagonal which is just $\mathbf{Q}_0$ (10.8). Thus the matrix **W** is really just a matrix representation, denoted by $\mathbf{Q_\omega}$ (below), of the body rate quaternion $\mathbf{q_\omega}$ (12.21), itself the quaternion form of the body rate vector $\boldsymbol{\omega}$ (which has no scalar component, i.e. $\omega_0 = 0$), and so the rate equation (12.22) can equally be written as the quaternion product

(12.26) $\dot{\mathbf{q}} = \dfrac{\mathbf{Q_\omega}}{2}\mathbf{q}$.

The matrices $\mathbf{W}$ and $\mathbf{Q_\omega}$ thus being one and the same thing

(12.27) $\mathbf{W} \equiv \mathbf{Q_\omega}$.

Just like $\mathbf{Q_q}$ (10.5), $\mathbf{Q_\omega}$ (or $\mathbf{W}$) can also be written in block matrix form as

(12.28) $\mathbf{Q_\omega} = \begin{pmatrix} 0 & -\boldsymbol{\omega}^T \\ \boldsymbol{\omega} & -\boldsymbol{\omega}\times \end{pmatrix}$.

where the 3x3, bottom right sub-matrix '$\boldsymbol{\omega}\times$' is the cross product representation matrix

(12.29) $\boldsymbol{\omega}\times = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$.

Note that the minus prefix on $\boldsymbol{\omega}\times$ in $\mathbf{Q_\omega}$ is intentional, as verified next.

Expanding out $\dot{\mathbf{q}}$ (12.26) in a vector form, using $\mathbf{Q_\omega}$ (12.28), just like (12.23), gives

(12.30) $\dot{\mathbf{q}} = q_0\boldsymbol{\omega} - \boldsymbol{\omega}^T\underline{\mathbf{q}} - \boldsymbol{\omega}\times\mathbf{q}$, and since $\mathbf{q}\times\boldsymbol{\omega} = -\boldsymbol{\omega}\times\mathbf{q}$ and $\underline{\mathbf{q}}^T\boldsymbol{\omega} = \boldsymbol{\omega}^T\underline{\mathbf{q}} = \underline{\mathbf{q}}\cdot\boldsymbol{\omega}$, then the two expressions for the rate, (12.23) and (12.30), are identical, as expected.

Given both $\mathbf{Q}_{\boldsymbol{\omega}}$ $(\equiv \mathbf{W})$ and $\mathbf{Q}_{\mathbf{q}}$ are just quaternion representation matrices, their eigenvalues and eigenvectors are a solved problem in URMT since they are both effectively a URM3 embedding of the URM3 Skew matrix $\boldsymbol{\omega}\times$ (12.29) into URM4. See AVE I, URM4 Skew (1-4), in particular the Skew form 2, **A** matrix (1.83). However, it is not so much the non-zero eigenvectors or eigenvalues of $\boldsymbol{\omega}\times$ that are of importance in the URMT formulation of angular dynamics and spin (next), but rather the fact that the quaternion rates can be obtained from either URMT's zero eigenvectors (I18) or its 'derivative operator' matrix $\mathbf{A}_+$ (6.2). These two aspects are explored in the next two sections on particle spin and rigid body dynamics, see particularly Sections (14-5) and (14-6) on zero eigenvectors and calculus using $\mathbf{A}_+$.

Appendices

24 Appendix (H) Rotations and Angular Dynamics

The content of this Appendix is relatively terse and serves merely as background material for the sections on spin and rigid body dynamics in the main text, Sections (13) and (14) respectively; see [13] for a full account of angular dynamics.

(H-1) Axes and Rotations

To discuss the angular dynamics of a rigid body, two sets of axes are required: 1) an inertial set fixed in space (or 'reference' or 'inertial' axes); 2) a set fixed in the rotating body, also known as 'body fixed axes'.

Both axes are orthogonal, right-handed sets.

The space axes are considered static and do not rotate, i.e. they are an inertial frame - they can move with constant velocity, just not rotate or accelerate.

Body axes are fixed in the body and rotate with it, i.e. they rotate with respect to space axes.

The body fixed axes are assumed aligned with the principal axes of inertia, i.e. the inertia tensor (or matrix) is entirely diagonal. If it has non-zero, off-diagonal elements then it can always be converted to diagonal form by a 'similarity transformation' [5].

The body x axis usually points forward along the line of longitudinal symmetry, e.g. for a cylinder this is along the axis of rotational symmetry - a rotation of the cylinder about this axis is known as a roll. For an aircraft, the x axis is along the body tube, pointing forward from the tail to the nose-cone, hence the aircraft rolls about this axis. Finally, for a spinning top, this is the axis pointing upward about which it spins.

(H-2) Transformation Matrices

The components of a vector $\mathbf{X}_a$, in an axes frame 'a', are related to those of a vector $\mathbf{X}_b$, in an axes frame 'b', by a linear (matrix) transformation $\mathbf{R}_{ab}$, i.e.

(H1) $\mathbf{X}_b = \mathbf{R}_{ab}\mathbf{X}_a$.

The subscript 'ab' on $\mathbf{R}_{ab}$ denotes a transformation from a to b. Some texts do this in reverse, i.e. 'ba' means a to b, but this is not the case herein.

The b axes are derived from a purely by means of rotations about one or more axes, both frames having a coincident origin, thus excluding translations (linear displacements) in the above transformation.

(H2)
Rotations can either be 'passive' or 'active'. Passive rotations are those just described, i.e. the axes are rotated and the vector stays fixed in space. Conversely, active rotations are those where the vector is rotated about some axis, and the axes themselves remain fixed in space. Only the former, passive

rotations are of interest herein, albeit going between the two types is little more than a sign change in the transformation (rotation angle) - see further, sub-section (H-4).

Positive rotations are counter clockwise when viewed looking along the rotation axis towards the origin, or clockwise when looking outward from the origin along the axis.

Consecutive rotations about different axes are not commutative, i.e. a rotation through angle ρ about the y axis, followed by a rotation through angle σ about the z axis does not give the same result as first rotating by σ about the z axis followed by a rotation ρ about the y axis. The order in which rotations are performed is therefore important, and there are at least two standards of ordering convention in common use. However, this text exclusively uses the aerospace convention 'Z-Y-X', as opposed to 'Z-X-Z', the latter more commonly given in theoretical texts [13].

(H-3) The Space-to-Body Axes Transformation

The general, 3D rotation comprises the following three individual rotations, in the order given, starting with the body and space axes coincident:

(H3)

(H3a) A yaw rotation through angle ψ about the z space axis

(H3b) A pitch rotation through angle θ about the new y axis

(H3c) A roll rotation through angle ϕ about the new x axis

For each rotation, the matrix transformation $\mathbf{R}$, that transforms the components of a vector in the space frame into components of the vector in the rotated frame, is given as follows:

1) Rotation yaw ψ about the z space axis

(H4) $\mathbf{R}_z(\psi) = \begin{pmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$, passive

2) Pitch θ about the new y axis, following the yaw rotation,

(H5) $\mathbf{R}_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$, passive

3) Roll ϕ about the new x axis, following the pitch rotation,

(H6) $\mathbf{R}_x(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix}$, passive

A rotation matrix is orthogonal, i.e. its inverse is its transpose (or vice-versa)

(H7) $\mathbf{R}^{-1} = \mathbf{R}^{T}$.

Each individual matrix also satisfies the following:

(H8) $\mathbf{R}^{T} = \mathbf{R}(-\rho), \rho \in \{\phi, \theta, \psi\}$,

so that the inverse of each matrix, using (H7), is thus also the same matrix with the sign of the angle inverted. Note that this is not the case when in combination (next), where both the sign of each angle and the order of consecutive transformations has to be reversed, which gives the equivalent of the transpose; see (H17) further below.

The complete 3-axis transformation is given by the following matrix product, strictly in the order shown:

(H9) $\mathbf{R}(\phi, \theta, \psi) = \mathbf{R}_x(\phi)\ \mathbf{R}_y(\theta)\ \mathbf{R}_z(\psi)$.

The combined matrix $\mathbf{R}(\phi, \theta, \psi)$ is known as the space to body transformation matrix, denoted by symbol $\mathbf{R}_{sb}$:

(H10) $\mathbf{R}_{sb}$ = space-to-body transformation matrix.

Multiplying out the three individual matrices in (H9) gives $\mathbf{R}_{sb}$ as

(H11) $$\mathbf{R}_{sb} = \begin{pmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\cos\theta \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\cos\theta \end{pmatrix}$$

(H12) Rotation matrices such as $\mathbf{R}_{sb}$ are also known as *direction cosine matrices* or 'DCM's for short.

(H13) The three angles ϕ , θ and ψ, roll, pitch and yaw in that order, are known as Euler angles, and uniquely specify the bodies orientation.

Euler angles are often given as the following ordered triple, which is the roll, pitch, yaw convention (an explanation follows)

(H14) $(\phi, \theta, \psi)\ \sim\ (roll, pitch, yaw)$.

The triple (ϕ, θ, ψ) is given in this ordered form since the angles roll, pitch and yaw are rotations about the x, y and z axis respectively, i.e. the ordering is in accordance with the ordered triple (x, y, z) . But be careful because this is not the order in which the rotations from space to body proceed, which is yaw (z) followed by pitch (y) followed by roll (x), as per (H3).

Whilst the triple (ϕ,θ,ψ) may appear as a vector quantity, it is most definitely not a vector. It cannot be added to another similar triple of Euler angles to give a physically meaningful result. Such a sum might be thought of when considering the attitude (H16) of a body with Euler angles (ϕ,θ,ψ), mounted on a platform that is, itself, orientated with Euler angles (ϕ',θ',ψ') with respect to an inertial, space frame. That is, the sum $(\phi+\phi',\theta+\theta',\psi+\psi')$ does not give the correct Euler angles of the body with respect to the space frame. The correct angles can be obtained by determining the matrix for the compound transformation from space-to-platform-to-body, and then using equations (H15) (below) to determine the Euler angles. However, they will not, in general, equal the sum of the two sets of Euler angles, except under very special conditions, most notably when all frames are rotated about a single axis only. This summation of angles is also valid (approximately anyhow) for small rotation angles, such rotations also then commute - see note (H41). The case of single axis rotations is actually used a great deal in this book's work on quaternions because it is a simple, illustrative case and, most importantly, is sufficient to deal with the concept of particle spin in URMT.

The Euler angles can be obtained from the elements of $\mathbf{R}_{sb}$ as follows, where $\mathbf{R}_{sb}(i,j)$ is the element in the ith row and jth column:

(H15)

$$\phi = \tan^{-1}(\mathbf{R}_{sb}(2,3)/\mathbf{R}_{sb}(3,3))$$

$$\theta = \sin^{-1}(-\mathbf{R}_{sb}(1,3))$$

$$\psi = \tan^{-1}(\mathbf{R}_{sb}(1,2)/\mathbf{R}_{sb}(1,1))$$

(H16)
Note that the term body 'attitude' is often thought of as the pointing angle of the x-axis in space, which is defined by just the two angles, pitch θ and yaw ψ. The roll angle ϕ doesn't affect the pointing, but simply the orientation of the body about what is usually its axis of rotational symmetry. Thus, if the body points in a fixed direction with respect to space axes, then its pitch and yaw angles remain constant, and the x-axis points in the same fixed direction. That said, such a two-angle definition of attitude is not assumed herein.

(H-4) The Body-to-Space Axes Transformation

To perform the reverse transformation, from body to space axes, the three rotations are performed in the reverse order with the sign of each of the angles inverted:

(H17)

Roll $-\phi$ (minus phi) about the body x axis

Pitch $-\theta$ (minus theta) about the new body y axis

Yaw $-\psi$ (minus psi) about the new body z axis.

The compound body-to-space transformation is thus given by

(H18) $\mathbf{R}_{bs} = \mathbf{R}_z(-\psi)\ \mathbf{R}_y(-\theta)\ \mathbf{R}_x(-\phi)$, (H19) $\mathbf{R}_{bs}$ = body-to-space transformation matrix

and this expands in full as

(H20) $$\mathbf{R}_{bs} = \begin{pmatrix} \cos\theta\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi \\ \cos\theta\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi \\ -\sin\theta & \sin\phi\cos\theta & \cos\phi\cos\theta \end{pmatrix}$$

Thus the components of a vector $\mathbf{r}'$, given in body axes, are transformed to components in space axes, vector $\underline{\mathbf{r}}$, by

(H21) $\underline{\mathbf{r}} = \mathbf{R}_{bs}\underline{\mathbf{r}'}$.

Note that $\mathbf{R}_{bs}$ is just the transpose of $\mathbf{R}_{sb}$

(H22) $\mathbf{R}_{bs} = \mathbf{R}_{sb}^T$.

Furthermore, since rotating from space to body axes and then back again through the same angles (albeit reversed) will return the same vector components as prior to the axes rotation, then the resulting compound transformation should be just the identity transformation, i.e. the 3x3 identity matrix $\mathbf{I}_3$ as given by

(H23) $\underline{\mathbf{r}} = \mathbf{R}_{bs}\underline{\mathbf{r}'} = \mathbf{R}_{bs}\mathbf{R}_{sb}\underline{\mathbf{r}} \Rightarrow \mathbf{R}_{bs}\mathbf{R}_{sb} = \mathbf{I}_3$.

Combining this result with (H22) then

(H24) $\mathbf{R}_{bs}\mathbf{R}_{sb} = \mathbf{R}_{sb}^T\mathbf{R}_{sb} = \mathbf{I}_3 \Rightarrow \mathbf{R}_{sb}^T = \mathbf{R}_{sb}^{-1}$,

confirming that $\mathbf{R}_{sb}$ is orthogonal in accordance with (H7).

Because $\mathbf{R}_{bs}$ is just the transpose of $\mathbf{R}_{sb}$ then the Euler angles can be obtained using (H15) above, but with transposed elements, i.e.

(H25)

$$\phi = \tan^{-1}(\mathbf{R}_{bs}(3,2)/\mathbf{R}_{bs}(3,3))$$

$$\theta = \sin^{-1}(-\mathbf{R}_{bs}(3,1))$$

$$\psi = \tan^{-1}(\mathbf{R}_{bs}(2,1)/\mathbf{R}_{bs}(1,1)).$$

There is, however, another more useful way to obtain the pitch and yaw from the vector components, in space axes, of a unit vector lying along the x-axis of the body, i.e. a vector $\underline{\mathbf{r}'}$ defined in body axes as

(H26) $\underline{\mathbf{r}}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $x' = 1$, $y' = 0$, $z' = 0$.

The components (x, y, z) of $\mathbf{r}$ in space axes are obtained from $\underline{\mathbf{r}}'$ as follows, using $\mathbf{R}_{bs}$ (H20), where the second and third column of $\mathbf{R}_{bs}$ are immaterial in the transformation, given y' and z' are both zero, hence blanked (hyphenated) out:

(H27) $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta\cos\psi & - & - \\ \cos\theta\sin\psi & - & - \\ -\sin\theta & - & - \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Expanding this transformation in terms of the components (x, y, z) of the unit vector, in space axes, gives

(H28)

$$x = \cos\theta\cos\psi$$

$$y = \cos\theta\sin\psi$$

$$z = -\sin\theta,$$

and from these then

(H29)

$$\theta = \sin^{-1}(-z)$$

$$\psi = \tan^{-1}(y/x)\ .$$

Note that this does not give the roll angle ϕ, merely the pointing direction of the x-axis. Albeit often the roll angle is immaterial and it is only the pitch and yaw angles that are required since these are sufficient to describe the pointing direction of the x axis.

The solution for the yaw ψ highlights a problem if the pitch angle is 90 deg (or an odd integer multiple $(2n+1)\pi/2$ radians), because then both x and y are zero and y/x is therefore indefinite. This is a notorious problem that arises because the roll and yaw are effectively ambiguous. Of course, having a pitch angle of exactly 90 deg is actually highly improbable in the real world of dynamics. Nevertheless, even when the pitch angle is close to 90 deg, there is another problem with the yaw rate, which hasn't yet been discussed. In this latter case, quaternions are invariably used instead of Euler angles for all intermediate calculations, and do not suffer from this problem. This issue with the yaw rate is discussed further below.

The above two expressions for pitch and yaw also give a simple geometric picture of the pointing direction. With z vertical, the pitch is actually the angle between the body x-axis and its projection on the x-y space plane, and, since it is the negative of z (H29), this means the angle is positive when the x-axis points below the x-y plane, in the opposite direction to positive z. However, in geodetic (Earth) applications, the z axis often points downward towards the geodetic centre of the earth (not quite the same as the geocentric centre, but never mind), so it is the negative z axis that points skyward, and hence a positive pitch is then the angle of the x axis above the x-y horizontal plane. In either case, positive z up or down, if the pitch is zero then the body x-axis lies in the x-y plane. The pitch angle is also sometimes referred to as elevation, albeit it may be the negative of the elevation depending on terminology and convention.

The yaw angle is simply the angle of how far the x-axis is rotated toward the y-axis before any pitch and roll rotations, i.e. (H3a) only. In geodetic axes, with z pointing down, then x is defined to point north, and y points east to complete a right-handed axes set. The yaw angle is then simply the angle clockwise from north, i.e. its '*northing*'. The yaw angle is often referred to as azimuth but, unlike pitch and elevation, usage of yaw and azimuth is not usually ambiguous. Nevertheless, caution is urged in all definitions, so always check the convention in use.

(H-5) Small Angle Approximations

Using the following, first order, small angle approximations for the trigonometric functions:

(H30) $\cos(\rho) \approx 1$, $\sin(\rho) \approx \rho$, $|\rho| << 1$, $\rho \in \{\phi, \theta, \psi\}$

then the small angle approximations of the rotation matrices (H4), (H5) and (H6) are

(H31) $$\mathbf{R}_x(\phi) \approx \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \phi \\ 0 & -\phi & 1 \end{pmatrix}, \ |\phi| << 1,$$

(H32) $$\mathbf{R}_y(\theta) \approx \begin{pmatrix} 1 & 0 & -\theta \\ 0 & 1 & 0 \\ \theta & 0 & 1 \end{pmatrix}, \ |\theta| << 1,$$

(H33) $$\mathbf{R}_z(\psi) \approx \begin{pmatrix} 1 & \psi & 0 \\ -\psi & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ |\psi| << 1.$$

Each of these can actually be written in the following form in terms of ' $\mathbf{J}$ ' matrices:

(H35) $$\mathbf{R}_x(\phi) \approx \mathbf{I}_3 + \phi \mathbf{J}_x, \ \mathbf{J}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

(H36) $$\mathbf{R}_y(\theta) \approx \mathbf{I}_3 + \theta \mathbf{J}_y, \ \mathbf{J}_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(H37) $$\mathbf{R}_z(\psi) \approx \mathbf{I}_3 + \psi \mathbf{J}_z, \ \mathbf{J}_z = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A small angle rotation of the axes through all three angles, in accordance with the rotation ordering in (H3), is thus given by the product:

(H38) $$\mathbf{R}_{sb} \approx (\mathbf{I}_3 + \phi \mathbf{J}_x)(\mathbf{I}_3 + \theta \mathbf{J}_y)(\mathbf{I}_3 + \psi \mathbf{J}_z),$$

and to first order this is just the same as the sum

(H39) $$\mathbf{R}_{sb} \approx (\mathbf{I}_3 + \phi \mathbf{J}_x + \theta \mathbf{J}_y + \psi \mathbf{J}_z)$$

which in full is

(H40) $$\mathbf{R}_{sb} \approx \begin{pmatrix} 1 & \psi & -\theta \\ -\psi & 1 & \phi \\ \theta & -\phi & 1 \end{pmatrix}.$$

This approximation of $\mathbf{R}_{sb}$ could, of course, also be obtained by applying the small angle approximations (H30) directly to matrix $\mathbf{R}_{sb}$ (H11).

(H41)
Notice that the sum (H39) does not depend upon the order the multiplications are performed in (H38), hence rotations are commutative to first order in the small angles.

(H-6) Euler Angle Rates and Body Rates

The angular rotation rate of a body is specified by a three-element vector $\underline{\boldsymbol{\omega}}$ whose components are the instantaneous angular rotation rates about each of the three, orthogonal, body-fixed axes

(H42) $$\underline{\boldsymbol{\omega}} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \equiv \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}.$$

Commonly, especially in aerospace, these three components $\omega_x, \omega_y, \omega_z$ (or $\omega_1, \omega_2, \omega_3$) are also denoted by the symbols p, q and r, i.e.

(H43)

$\omega_x, \omega_1 \equiv p =$ body roll rate about the x axis

$\omega_y, \omega_2 \equiv q =$ body pitch rate about the y axis

$\omega_z, \omega_3 \equiv r =$ body yaw rate about the z axis

$$\underline{\boldsymbol{\omega}} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

Notes

The symbol p does not correspond to the p in pitch, neither does symbol r correspond to the r in roll. Although confusing, the reason is simply that the ordered triple (p, q, r) is alphabetically ordered and corresponds with the axes ordering (x, y, z), which is roll-pitch-yaw.

The components $\omega_x, \omega_y, \omega_z$ are used instead of $\omega_1, \omega_2, \omega_3$ from here onward, simply to remove redundancy - both forms are identical (H42).

The body rates p, q and r are not the same as the URMT dynamical variables, P, Q and R, although the notation is not unrelated, as it was realised early in the development of URM3 [1] that the matrix and its velocity (angular velocity) properties were not dissimilar. Neither are the body rates generally the same as the Euler angle rates $\dot{\phi}, \dot{\theta}, \dot{\psi}$, the only exception being when the rotation is about a single axis.

(H44)

$$\omega_x = p \neq \dot{\phi}$$

$$\omega_y = q \neq \dot{\theta}$$

$$\omega_z = r \neq \dot{\psi}$$

In fact, $\underline{\boldsymbol{\omega}}$ is related to the Euler angle rates $\dot{\phi}, \dot{\psi}, \dot{\theta}$ by the following matrix transformation equations:

(H45) $$\underline{\boldsymbol{\omega}} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix} + \mathbf{R}_x(\phi) \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix} + \mathbf{R}_x(\phi)\mathbf{R}_y(\theta) \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}.$$

Substituting for $\mathbf{R}_x(\phi)$ (H6) and $\mathbf{R}_y(\theta)$ (H5) gives the components of $\underline{\boldsymbol{\omega}}$ as

(H46)

$$\omega_x = p = \dot{\phi} - \dot{\psi} \sin\theta$$

$$\omega_y = q = \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi$$

$$\omega_z = r = -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi,$$

Alternatively, in matrix form,

(H47) $$\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} \equiv \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \cos\theta\sin\phi \\ 0 & -\sin\phi & \cos\theta\cos\phi \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}.$$

To obtain the Euler rates in terms of the body rates, it is simplest to pre-multiply $\underline{\boldsymbol{\omega}}$ (H45) by the inverse of $\mathbf{R}_x(\phi)$, which is equivalent to 'unrolling' the axes, i.e. a rotation of $-\phi$ about the body fixed x axis (not the original space fixed x axis) as in

(H48) $\mathbf{R}_x^{-1}(\phi) = \mathbf{R}_x(-\phi) = \mathbf{R}_x^T(\phi)$.

Using the odd and even function properties of the trigonometric functions, i.e. $\sin(-\phi) = -\sin\phi$ and $\cos(-\phi) = \cos\phi$, then the inverse of $\mathbf{R}_x(\phi)$ (H6) is thus

(H49) $$\mathbf{R}_x^{-1}(\phi) = \mathbf{R}_x(-\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

Multiplying $\underline{\boldsymbol{\omega}}$ (H45) throughout by $\mathbf{R}_x(-\phi)$ gives

(H50) $$\mathbf{R}_x(-\phi)\underline{\boldsymbol{\omega}} = \mathbf{R}_x(-\phi)\begin{pmatrix} \dot\phi \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \dot\theta \\ 0 \end{pmatrix} + \mathbf{R}_y(\theta)\begin{pmatrix} 0 \\ 0 \\ \dot\psi \end{pmatrix},$$

and substituting for $\mathbf{R}_x(-\phi)$ (H49) and $\mathbf{R}_y(\theta)$ (H5) gives

(H51)

$$\omega_x = \dot\phi - \dot\psi\sin\theta$$

$$\omega_y\cos\phi - \omega_z\sin\phi = \dot\theta$$

$$\omega_y\sin\phi + \omega_z\cos\phi = \dot\psi\cos\theta.$$

These equations can easily be rearranged to give the Euler rates in terms of the body rates as

(H52)

$$\dot\phi = \omega_x + (\omega_y\sin\phi + \omega_z\cos\phi)\tan\theta$$

$$\dot\theta = \omega_y\cos\phi - \omega_z\sin\phi$$

$$\dot\psi = (\omega_y\sin\phi + \omega_z\cos\phi)/\cos\theta,$$

where the first expression for $\dot\phi$ uses the third expression for $\dot\psi$.

These are alternatively written in terms of p, q, r as

(H53)

$$\dot\phi = p + (q\sin\phi + r\cos\phi)\tan\theta$$

$$\dot{\theta} = q\cos\phi - r\sin\phi$$

$$\dot{\psi} = (q\sin\phi + r\cos\phi)/\cos\theta\,,$$

and in matrix form as

(H54) $$\begin{pmatrix}\dot{\phi}\\ \dot{\theta}\\ \dot{\psi}\end{pmatrix} = \begin{pmatrix}1 & \sin\phi\tan\theta & \cos\phi\tan\theta\\ 0 & \cos\phi & -\sin\phi\\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}.$$

Both the Euler roll and yaw rate, $\dot{\phi}$ and $\dot{\psi}$, show the issue highlighted earlier, i.e. when the pitch angle θ is 90 deg it involves a divide by zero ($\cos\theta$ is zero and $\tan\theta$ is infinite). This is probably the biggest reason cited to use quaternions in place of Euler angles since quaternion rates can be integrated through a 90 deg pitch angle whilst remaining finite.

The physical explanation for this singularity is fairly straightforward with reference to, for example, a near-vertical spinning top. Suppose, the body x axis (about which it spins) points near vertical with a pitch angle of 89 deg, i.e. almost aligned with the z space axis, and with a yaw angle of 0 deg, then should the axis go from 89 to 91 deg via 90 deg then its yaw angle becomes 180 deg. Thus the small movement in pitch from 89 to 91 deg has created a large 180 deg change in yaw (azimuth). The result, is a very high yaw rate compared with the pitch rate, i.e. a 180 deg/s yaw rate versus a 2 deg/s pitch rate - assuming the manoeuvre occurs in 1s. Note that the pitch movement from 89 deg to 91 deg, going through 90 deg, is known as 'crossing the zenith' - particularly in astronomy. Of course, spinning tops do not actually go through the zenith but tend to precess around it. Nevertheless, telescopes or radar antenna, when tracking an object or changing to track a different object, may well frequently cross the zenith. They will use the azimuth rate in their tracking loop and therefore experience a large yaw rate near the zenith. At the zenith (90 deg pitch), the yaw rate, $\dot{\psi}$ (H53), will blow-up to infinity, and so too the roll rate $\dot{\phi}$. Thus, the 90 deg pitch singularity can be a big problem in real-world applications, and is resolved by using quaternions, which do not suffer this deficiency. Note that some applications do not track in roll, e.g. steering a telescope is in yaw and pitch only (azimuth and elevation ~ right-ascension and declination), hence the emphasis here is on yaw, not roll.

(H-7) Euler's Angular Equations of Motion

Equations (H54) are commonly used to obtain the Euler angle rates $\dot{\phi},\dot{\theta},\dot{\psi}$ from the vector body rates $\omega_x,\omega_y,\omega_z$ (or p,q,r) because the body rates are obtained by integration of the angular accelerations $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$, which are themselves calculated straight from the following force/torque (Γ_x, Γ_y, Γ_z) equations, also known as Euler's equations:

(H55)

$$\Gamma_x = I_{xx}\dot{\omega}_x - \omega_y\omega_z(I_{yy} - I_{zz})$$

$$\Gamma_y = I_{yy}\dot{\omega}_y - \omega_z\omega_x(I_{zz} - I_{xx})$$

$$\Gamma_z = I_{zz}\dot{\omega}_z - \omega_x\omega_y(I_{xx} - I_{yy})\,.$$

The three inertia components I_{xx}, I_{yy} and I_{zz} are about the principal axes, and obtained from the diagonalised inertia tensor $\mathbf{I}_b$, i.e.

(H56) $\mathbf{I}_b = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$.

The above Euler equations are trivially rearranged to give the angular accelerations $\dot{\omega}_x$, $\dot{\omega}_y$, $\dot{\omega}_z$ as

(H57)

$$\dot{\omega}_x = \frac{\Gamma_x}{I_{xx}} + \omega_y \omega_z (\frac{I_{yy} - I_{zz}}{I_{xx}})$$

$$\dot{\omega}_y = \frac{\Gamma_y}{I_{yy}} + \omega_z \omega_x (\frac{I_{zz} - I_{xx}}{I_{yy}})$$

$$\dot{\omega}_z = \frac{\Gamma_z}{I_{zz}} + \omega_x \omega_y (\frac{I_{xx} - I_{yy}}{I_{zz}}),$$

and written alternatively in terms of symbols p, q, r as

(H58)

$$\dot{p} = \frac{\Gamma_x}{I_{xx}} + qr(\frac{I_{yy} - I_{zz}}{I_{xx}})$$

$$\dot{q} = \frac{\Gamma_y}{I_{yy}} + rp(\frac{I_{zz} - I_{xx}}{I_{yy}})$$

$$\dot{r} = \frac{\Gamma_z}{I_{zz}} + pq(\frac{I_{xx} - I_{yy}}{I_{zz}}).$$

The complete process to go from the known torques acting, to obtain the Euler angles and thus the body orientation, is an extremely common application in aerospace, the motor industry, robotics and computer graphics, and invariably involves quaternions, all detailed in the main text; see Sections (9) to (12) on quaternions, and Section (14) on rigid body dynamics.