Compactification of an n-dimensional eigenvector space over long evolutionary timescales.

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Abstract

This paper shows how a discrete, n-dimensional eigenvector space can appear of lower dimension over long evolutionary timescales, ultimately compactifying to appear as a two-dimensional subspace within three dimensions.

Each excess dimension, i.e. any dimension higher than the third, has a unique, temporal coordinate, not necessarily associated with the familiar laboratory time, which controls the evolution of the dimension. Over a long evolutionary period, in a particular excess dimension, the dimension expands relative to all other excess dimensions, but appears to contract relative to the first three dimensions. Specifically, the entire n-dimensional space appears to align along a particular direction in the three-dimensional space, the direction given by one of the eigenvectors, which is physically associated with an acceleration vector, and specified by two, non-temporal, arbitrary parameters. The third parameter in the three-dimensional space is a temporal coordinate, which also controls the evolution of the three dimensions, and shows the same alignment behaviour as for the excess dimensions. The initial state of the entire space is specified by the initial values for the acceleration vector and, most importantly, a single energy-related constant controls the initial size of all excess dimensions.

The paper mathematically details the compactification process by way of a four and five-dimensional case, expressed in terms of the three-dimensional solution, with a full 5D numeric example provided in the Appendices. A complete n-dimensional solution is given and the compactification arguments generalised for an arbitrary number of dimensions.

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Acronyms

DCE Dynamical Conservation Equation
gcd greatest common divisor
URMT Unity Root Matrix Theory
URMn the $n \times n$ matrix formulation of URMT
STR Special Theory of Relativity

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1 Introduction

A discrete[1], n-dimensional, vector space[2] can be generated from the eigenvectors of a unity root matrix, as first described in [1] for the three-dimensional case; see also [2], [3] for an extensive, freely available, PDF overview. Recent work [4] extends the three-dimensional case to four dimensions and beyond, driven primarily by physical requirements to obtain a proper, four-dimensional space-time and non-zero, relativistic intervals. From [4], it is evident that URMT actually generalises to any number of dimensions 'n', for square, $n \times n$ matrices, whilst retaining all the physical features present in URM3, including conservation equations, related scalar invariants and a ternary nature, not least, its three-dimensional eigenvectors.

The three-dimensional form of URMT, known as URM3 and summarised in Appendix (A), is a completely solved problem under special 'Pythagoras conditions' (I9), with the unity root matrix $\mathbf{A}_{30}$ (A2b) producing three eigenvectors, $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ (A8), for eigenvalues $\lambda = +C$ (footnote[3]), $\lambda = 0$ and $\lambda = -C$ respectively, where the two eigenvectors, $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$, are Pythagorean triples, and the third, $\mathbf{X}_{30}$, satisfies a hyperbolic 'Dynamical Conservation Equation' (DCE), (5.7). Appendix (A) provides a complete analytic solution to URM3 with an example, $\mathbf{X}_{3+} = (4,3,5)$, also included.

The relevance of the DCE[4] here is that the formulation of URMT can be expressed starting with this as an assumed conservation equation in the dynamical variables, for conserved quantity C^2. By applying a form of transformation invariance, both its dynamical equations and solutions can be obtained, [1],#1[5]. This invariance[6] generates a single, global variational

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[1] URMT is currently formulated entirely in integers (14.2) and, hence, if it is a physical description of nature, it is a discrete description, which probably operates at the Planck scale upward, see [1] or [3]. However, this is speculative and a definitive scale is yet to be decided.

[2] An infinite set (space) of n linearly independent eigenvectors might be a preferable description for the purists, rather than 'vector space'. The space is not generally closed and neither is there a zero vector - although this could be added. See [5] for a strict definition of a vector space. The URM3 set of eigenvectors (space) is defined by the 'lattice' in [1] and [3]. The key point is that every vector in the lattice is an eigenvector of the unity root matrix and arbitrary, linear combinations of eigenvectors do not generally give another eigenvector in the lattice, as is true for any general set of eigenvectors for distinct eigenvalues. That is not to say the n URMT eigenvectors cannot form the basis of an n-dimensional vector space; indeed they can by their linear independence, which is why the term 'vector space' is used loosely. Currently, however, URMT focuses solely on the eigenvectors themselves, but not arbitrary functions of them, such as linear combinations.

[3] This is big 'C', a fundamental constant in URMT and not to be confused with little c, the speed of light. Albeit, big C also has a physical interpretation as a velocity constant and, more so, C^2 has an interpretation as that of energy (per unit mass), see (2.1). Big C was originally chosen as the first letter of the word 'Constant'. That it appears to be remarkably similar to little c in physical nature is purely coincidental but, admittedly, this will take some faith.

[4] The DCE (5.7) is also the singularity condition for the eigenvector matrix equation, i.e. $\det(\mathbf{A}_{30} - C\mathbf{I}_3) = 0$, where $\mathbf{I}_3$ is the usual 3×3 identity matrix.

[5] Notation '[1],#N' denotes paper number N, N=1..6, in [1].

[6] The term 'invariance' manifests itself in the static nature of the URM3 eigenvector $\mathbf{X}_{3+}$, which is not a function of any evolutionary parameter, whereas the other two URM3 eigenvectors, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$, are explicit functions of the URM3 evolutionary time, i.e. temporal parameter t_3.

parameter that can be physically associated with time 't_3' [7] (symbol m or δ in [1] to [3]) and, by varying this time, the evolution of the 3D eigenvector space can also be studied.

Using a similar form of transformation invariance to URM3 (above), each excess (I4), jth dimension, $j = 4 \ldots n$, (URMn), is also attributed a unique temporal parameter t_j. This parameter appears in the n-dimensional eigenvector solution and, therefore, the evolution of any specific, jth dimension can be independently varied by varying t_j. In particular, by studying the behaviour of one or more dimensions for large evolutionary periods, the apparent relative contraction of all excess dimensions can be observed, i.e. compactification.

As regards the first three dimensions, i.e. those of URM3, their evolution with respect to t_3 (Appendix (B)), is well characterised in URMT and, for long evolutionary periods, i.e. large t_3, the three-dimensional eigenvector space of URM3 is seen to 'flatten' (or align[8]). The concept of 'flattening' (I5) is described in [1],#3 and [3], and refers to the fact that the two eigenvectors $\mathbf{X}_{3-}$ and $\mathbf{X}_{30}$ align anti-parallel[9] to $\mathbf{X}_{3+}$ as URM3 evolves, i.e. as t_3 increases.

In fact, exactly the same behaviour is seen to occur for all excess dimensions, i.e. the entire vector space aligns with the URM3 eigenvector $\mathbf{X}_{3+}$, as will be shown in this paper. This vector is parameterised by two arbitrary integers k and l (see footnote 7), and forms a discrete cone[10] surface in three dimensions. Hence compactification can be thought of as stopping at the 2D conical surface residing in the three-dimensional eigenvector space of URM3.

Because URM3 is a fully solved and documented mathematical problem (see any of [1] to [3]), only the compactification behaviour of the excess dimensions, with respect to the 3D world of URM3, is studied.

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[7] Whilst the URM3 analytic solution is parameterised by a temporal parameter t_3, it is completely specified by two additional, arbitrary, integer parameters k and l. However, only t_3 is of a true, temporal nature, and k and l have units of $\left(LT^{-2}\right)^{1/2}$, i.e. the square root of acceleration (eigenvector $\mathbf{X}_{3+}$), and only ever appear in expressions of the second degree, e.g. $x = 2kl$, see Appendix (A3). Suffice to note, the three eigenvectors $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ are parameterised by all three parameters k, l and t_3.

[8] It is an alignment in that the vectors converge to align in the direction of the single vector $\mathbf{X}_{3+}$, which is static and invariant to arbitrary variations in any evolutionary parameter. However, $\mathbf{X}_{3+}$ is actually a two-parameter family of integer vectors (footnote 7), hence a 2D discrete subspace of 3D and, in this sense, the 3D flattens (I5) to 2D, see also footnote (10).

[9] That they align anti-parallel, and not parallel, is largely a choice of sign convention.

[10] See [1],#3 and [3] regarding URM3 geometry, cones, hyperboloids and a lattice.

2 A Standard Physical Interpretation

Since the mathematics of URMT is thought to have strong links to the subject of 'Physics in Integers', a standard physical interpretation[11] (or association) is ascribed to all variables, see [1],#3, and [3]. A brief summary of the physical associations follows:

(2.1) $\mathbf{X}_{3+}$, x, y, z, acceleration or force per unit mass, LT^{-2}

$\mathbf{A}_{30}$, $\mathbf{X}_{30}$, P, Q, R, C, velocity[12] or momentum per unit mass, LT^{-1}

$\mathbf{X}_{3-}$, position, L

t_3, m, δ, time, T

C^2, velocity squared or total energy E ($E = C^2$) per unit mass, L^2T^{-2}.

The same interpretation generalises to all higher, n-dimensional quantities: vectors $\mathbf{X}_{n+}$, $\mathbf{X}_{n-}$, $\mathbf{X}_{n0}$ ($\mathbf{X}_{n0A}$, $\mathbf{X}_{n0B}$, …) and their elements; matrix $\mathbf{A}_{n0}$ and its elements (the dynamical variables); and evolutionary parameter t_n, using the following notational equivalents[13]

(2.2) $\mathbf{X}_{3+} \sim \mathbf{X}_{n+}$, acceleration

$\mathbf{A}_{30} \sim \mathbf{A}_{n0}$, $\mathbf{X}_{30} \sim \mathbf{X}_{n0A}$, $\mathbf{X}_{n0B}$, $\mathbf{X}_{n0C}$, … velocity

$\mathbf{X}_{3-} \sim \mathbf{X}_{n-}$, position

$t_3 \sim t_n$ time

C^2, as above.

The paper now proceeds to the mathematical formulation with which the compactification behaviour can be demonstrated.

[11] This is not the only physical interpretation but it currently seems the best as regards Physics.

[12] The elements of all unity root matrices, e.g. $\mathbf{A}_{30}$ with elements P, Q, R, are termed 'dynamical variables' as they can be physically associated with velocity, or momentum per unit mass.

[13] The URM3 notation has been embellished in this paper to differentiate its vectors and matrices from those of URM4 and beyond. All URM3-specific variables are now subscripted with a '3', as are all URM n variables subscripted with an n.

3 The General URM5 Formulation

The most general form of URMT, i.e. the n-dimensional 'URMn', is defined for a single, square $n \times n$ matrix $\mathbf{A}$ and vector $\mathbf{X}$, which is an eigenvector to matrix $\mathbf{A}$ for eigenvalue C, i.e. $\mathbf{AX} = C\mathbf{X}$. Specifically, for URM5, the unity root[14] matrix $\mathbf{A}$ [15] is defined as follows:

(3.1) $$\mathbf{A} = \begin{pmatrix} 0 & M & H & N & J \\ \overline{M} & 0 & S & T & U \\ \overline{H} & \overline{S} & 0 & R & \overline{Q} \\ \overline{N} & \overline{T} & \overline{R} & 0 & P \\ \overline{J} & \overline{U} & Q & \overline{P} & 0 \end{pmatrix}$$, (footnote [16])

comprising ten dynamical variables

(3.2) $M, H, N, J \in \mathbb{Z}$
$S, T, U \in \mathbb{Z}$
$P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0)$, URM3 (A1b)

and their conjugates

(3.3) $\overline{M}, \overline{H}, \overline{N}, \overline{J} \in \mathbb{Z}$
$\overline{S}, \overline{T}, \overline{U} \in \mathbb{Z}$
$\overline{P}, \overline{Q}, \overline{R} \in \mathbb{Z},\ (\overline{P}, \overline{Q}, \overline{R}) \neq (0,0,0)$. (footnote [17]), URM3 (A1c)

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[14] Only the URM3 dynamical variables P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ are true integer, 'unity roots' (A13) and, only then, when eigenvalue $C = 1$ (3.5). Otherwise, they are generally known as power-residues [6]. Both forms are isomorphic to the complex roots of unity, e.g. $P \sim Z$, $\overline{P} \sim Z^*$, for unity eigenvalue. The unity root aspect is not required in this paper but Appendix (A) provides some background detail at the end (A13).

[15] The $\mathbf{A}$ matrix naturally embeds the URM3, 3x3 matrix $\mathbf{A}_{30}$ (A2b)

[16] The usage of four, non-consecutive capitals, M, H, N, J in the top row and left column of $\mathbf{A}$ is unfortunate, but primarily due to the inability to find four such consecutive capitals that are not already reserved in URMT. The peculiar alphabetic ordering, i.e. M, H, N, J, is also legacy and due to some other unpublished simplifications to the matrix. A similar issue arises with the commonly used Pythagorean triple (3,4,5), which in URMT is ordered (4,3,5). Mathematically they are, of course, quite distinct, although URMT covers this, see [2].

[17] Conjugates, such as $\overline{P}, \overline{Q}, \overline{R}$, are linked to their standard forms P, Q, R by conjugate relations [2], which are equivalent to the Pythagoras conditions (4.2c).

A single eigenvector $\mathbf{X}$ is defined comprising five coordinates v, w, x, y, z
(3.4)

(3.4a) $$\mathbf{X} = \begin{pmatrix} v \\ w \\ x \\ y \\ z \end{pmatrix}$$

(3.4b) $v, w, x, y, z \in \mathbb{Z},\ (v, w, x, y, z) \neq (0,0,0,0,0)$
(3.4c) $\gcd(v, w, x, y, z) = 1$ (footnote [18])

which, as stated, is an eigenvector of matrix $\mathbf{A}$, eigenvalue C [19]

(3.5) $\mathbf{AX} = C\mathbf{X}$, $C \in \mathbb{Z},\ C \geq 1$ (see also footnotes [20] and [21])

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[18] The gcd condition (3.4c) is primarily imposed to uniquely specify the elements (URM3 scale factors $\alpha_3, \beta_3, \gamma_3$ (A6)) for another eigenvector $\mathbf{X}_-$ of $\mathbf{A}$, eigenvalue $-C$. See also Appendix (C15), (C16).

[19] The eigenvalue C is of physical importance to URMT since it can be associated with a velocity and it appears of second degree (quadratic) in scalar invariants, i.e. equivalent to the kinetic energy per unit mass. Most notably, C^2 is the conserved quantity in the Dynamical Conservation Equation (5.7), also appearing in numerous other scalar invariants.

[20] Once a non-zero eigenvalue C is imposed, the URM3 dynamical variables P, Q, R cannot all be trivially zero, i.e. $(P, Q, R) \neq (0,0,0)$ (3.2). By conjugate relations (footnote 17), neither are $\overline{P}, \overline{Q}, \overline{R}$ all zero (3.3). One or two of the three P, Q, R can possibly be zero, but not all three simultaneously. This also constrains the elements v, w, x, y, z of vector $\mathbf{X}$ in the same way, (3.4b). That no vector in URMT can comprise all zeros, and therefore have zero magnitude, is an algebraic consequence of mandating $C \geq 1$. Effectively, it means URMT has no singularities and, naturally, this is a highly desirable physical attribute.

[21] URM3 is generally solved for a unity eigenvalue, $C = 1$, and the solution can then be used to obtain the solution for arbitrary eigenvalue, $C > 1$, see [1],#6. The unity value is considered to be the normalised form of URMT, it also makes for a true definition of the dynamical variables as unity roots; see Appendix (A13).

4 Simplifications and Pythagoras Conditions

As for all forms of URMT, i.e. URMn, $n \geq 3$, the general case requires simplifying conditions to produce analytic solutions that are considered to be of physical relevance. There is a common set of conditions, termed 'Pythagoras conditions' (I9), that are present in every incarnation of URMn[22], and where the elements of every eigenvector $\mathbf{X}$, for a non-zero eigenvalue ($\pm C$), obey the Pythagoras equation, i.e.

(4.1) $\quad 0 = v^2 + w^2 + x^2 + y^2 - z^2$.

Because of the complexity and goals of URM5, it is currently only studied under Pythagoras conditions, noting that there are also a few additional conditions imposed to obtain some specific, highly desirable physical properties, e.g. an 'invariant, zero Potential', also present in URM3 and URM4.

All work hereafter will assume to be formulated under Pythagoras conditions.

URM5 Pythagoras Conditions

The URM5 Pythagoras conditions on the conjugate dynamical variables (3.3) are

(4.2)
(4.2a) $\overline{M} = -M$, $\overline{H} = -H$, $\overline{N} = -N$, $\overline{J} = J$
(4.2b) $\overline{S} = -S$, $\overline{T} = -T$, $\overline{U} = U$
(4.2c) $\overline{P} = P$, $\overline{Q} = Q$, $\overline{R} = -R$.

When under these conditions, matrix $\mathbf{A}$ is relabelled $\mathbf{A}_{50}$ (footnote [23]) where the '5' in the subscript denotes URM5, and the '0' represents the standard form of the unity root matrix under Pythagoras conditions[24]. From here onward, all matrices and eigenvectors are

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[22] All Pythagoras conditions for URMn include URM$(n-1)$ as a subset.

[23] In URM3, under a general, unified scheme [1],#5, there are actually three such matrices, $\mathbf{A}_0$, $\mathbf{A}_+$ and $\mathbf{A}_-$, albeit only $\mathbf{A}_0$ (~$\mathbf{A}_{50}$ here, and $\mathbf{A}_{n0}$ in general) is used explicitly in this paper.

[24] Regardless of the actual values of the elements of $\mathbf{A}_{50}$ (4.3), any eigenvector $\mathbf{X}$, for non-zero eigenvalue, actually satisfies the Pythagoras equation (4.1), e.g. for URM5 the eigenvector $\mathbf{X}_{5+}$ is a trivial Pythagorean quintuple $(0,0,x,y,z)$ as in $0 = 0^2 + 0^2 + x^2 + y^2 - z^2$. The other non-zero eigenvalue of note herein is $-C$, the others all being zero, see (5.9). The associated eigenvector for the non-zero eigenvalue $-C$ is $\mathbf{X}_{5-}$ (11.1b), and is a non-trivial Pythagorean quintuple. It is no coincidence that the elements of the eigenvectors $\mathbf{X}_{5+}$ and $\mathbf{X}_{5-}$ satisfy the Pythagoras equation (admittedly $\mathbf{X}_{5+}$ is really just a 3D embedding, see (5.17)). It was a goal of URMT to pursue such eigenvectors as they have a Minkowski form. It is actually eigenvector $\mathbf{X}_{4-}$ (8.1b), (URM4), with a physical interpretation (2.1) as a position vector, that is considered to have its fourth-element 'γ_4' as the more familiar ct in the STR four-vector position $(x \quad y \quad z \quad ct)$. Furthermore, and most importantly, the biggest reason to extend URMT to four and five dimensions was to specifically incorporate three spatial dimensions, one time dimension, and also allow for non-zero intervals $c\tau$, i.e. proper

subscripted with an 'n' according to which n-dimensional incarnation of URMn they represent.

Using conditions (4.2), matrix $\mathbf{A}$ (3.1) becomes $\mathbf{A}_{50}$:

$$(4.3)\quad \mathbf{A}_{50} = \begin{pmatrix} 0 & M & H & N & J \\ -M & 0 & S & T & U \\ -H & -S & 0 & R & Q \\ -N & -T & -R & 0 & P \\ J & U & Q & P & 0 \end{pmatrix}.$$

A kinetic term K and Potential term V are defined as follows, whereby these forms are intentionally chosen to simplify the characteristic equation for $\mathbf{A}_{50}$

$$(4.4)\quad K = J^2 + P^2 + Q^2 + U^2 - (H^2 + M^2 + N^2 + R^2 + S^2 + T^2)$$

$$(4.5)\quad \begin{array}{l} V = [QT-(PS+RU)]^2 + [NQ-(JR+HP)]^2 + \\ [HU-(JS+MQ)]^2 + [NU-(JT+MP)]^2 - \\ [HT-(MR+NS)]^2 \end{array}$$, footnote [25]

Using these two terms, K and V, the characteristic equation, for matrix $\mathbf{A}_{50}$, eigenvalue λ, is

$$(4.6)\quad 0 = \lambda(-\lambda^4 + K\lambda^2 + V).$$

time τ as in $-(c\tau)^2 = x^2 + y^2 + z^2 - (ct)^2$. Keep in mind that the URM4, four-vector $\mathbf{X}_{4-}$ (8.1b) is the positional equivalent of the STR four-vector $(x \quad y \quad z \quad ct)$ since the four-vector $\mathbf{X}_{4+} = (w \quad x \quad y \quad x)$ is physically associated with an acceleration vector (2.2). Albeit, $\mathbf{X}_{4-}$ and $\mathbf{X}_{4+}$ are 'dual' to each other in URMT and the entire theory can be formulated in one or the other. This 'duality' is discussed, for URM3, in [1],#3,#5 and [3].

[25] The form of the Potential (4.5) is specific to each incarnation of URMn. Under URM3 Pythagoras conditions it is always zero. In URM4 it is given by just the first term in (4.5), i.e. $V = [QT-(PS+RU)]^2$, and is not generally zero under URM4 Pythagoras conditions without further simplifications, see [4].

5 Invariant Zero Potential Conditions

With the specific goal of obtaining two symmetric, non-zero eigenvalues, with all others zero, i.e. $\lambda = \pm C,0,0,0$ for URM5 (5.9), it is desirable to make the Potential V zero (4.5). Not only that, but keep it zero for arbitrary evolutionary times t_j, $j = 3 \ldots n$, which is termed an invariant, zero Potential in URMT.

An invariant Potential (not specifically zero) is also very important in URMT physics because, as in any energy conservation equation, it means the kinetic energy is a constant. In this case, the DCE (5.7) has a constant kinetic term K equal to the total energy E, i.e. $E = C^2 = K$. It also means there is no kinetic/Potential energy interchange and, hence, no force, or at least no forces that do any work. In brief, it makes a good simple case to study.

With these factors in mind, and with the benefit of a lot of hindsight, explained in [4], the additional conditions for such a zero, invariant Potential, $V = 0$, are as follows, with some explanation given shortly after:

The first two coordinates of $\mathbf{X}_{5+}$ are zeroed, i.e.

(5.1)
(5.1a) $v = 0$, $w = 0$,

and the URM4 and URM5 dynamical variables, S,T,U and M,H,N,J respectively, are assigned as scalar multiples of the eigenvector $\mathbf{X}_{5+}$ (3.4a), now with two zero coordinates v,w (above), where the scalars are the evolutionary parameters t_4 and t_5,

(5.1b) $M = 0$, $H = -t_5 x$, $N = -t_5 y$, $J = +t_5 z$
(5.1c) $S = -t_4 x$, $T = -t_4 y$, $U = +t_4 z$
(5.1d) $t_4, t_4 \in \mathbb{Z}$.

With these conditions (5.1), the matrix $\mathbf{A}_{50}$ and eigenvector $\mathbf{X}$ (now relabelled $\mathbf{X}_{5+}$) become

(5.2) $$\mathbf{A}_{50} = \begin{pmatrix} 0 & 0 & -t_5 x & -t_5 y & +t_5 z \\ 0 & 0 & -t_4 x & -t_4 y & +t_4 sz \\ +t_5 x & +t_4 x & 0 & R & Q \\ +t_5 y & +t_4 y & -R & 0 & P \\ +t_5 z & +t_4 z & Q & P & 0 \end{pmatrix}, \quad \mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ x \\ y \\ z \end{pmatrix}$$, (footnote [26]),

with eigenvector equation

[26] The eigenvector $\mathbf{X}_{5+}$, whilst five-dimensional, evidently only occupies the subspace of URM3 vector $\mathbf{X}_{3+}$. This is, of course, intentional. Non-trivial (I7) 4D and 5D vectors, i.e. those with four and five, non-zero elements respectively, will emerge as the work progresses; they are the eigenvectors to the other eigenvalues (5.9).

(5.3) $\mathbf{A}_{50}\mathbf{X}_{5+} = C\mathbf{X}_{5+}$.

A justification for the form of $\mathbf{A}_{50}$ and $\mathbf{X}_{5+}$, for the general, n-dimensional version, is given in Appendix (C).

An explanation of conditions (5.1) follows shortly after a quick summary of their effect on the energy terms and eigenvalues.

Every bracketed term in the Potential V (4.5) is now zero and so too, therefore, the overall sum

(5.4) $V = 0$.

Substituting for the dynamical variables M, H, N, J, S, T, U from (5.1) into the kinetic term K (4.4), and using the Pythagoras equation (4.1), gives

(5.5) $K = P^2 + Q^2 - R^2$.

By associating K with the positive constant C^2, i.e.

(5.6) $K = C^2$,

then, for a zero Potential (5.4), the kinetic expression (5.5) becomes the familiar URM3 Dynamical Conservation Equation (DCE), as per URM3 Pythagoras conditions

(5.7) $C^2 = P^2 + Q^2 - R^2$, the DCE.

With a zero Potential (5.4), and a kinetic term (5.6), the characteristic equation (4.6) becomes

(5.8) $0 = \lambda^3(C^2 - \lambda^2)$.

This characteristic equation factors with the following five eigenvalues as roots, three of which are zero

(5.9) $\lambda = \pm C, 0, 0, 0$, (footnote [27]).

An explanation on the choice of conditions (5.1), and the rather abstract form of $\mathbf{A}_{50}$ (5.2), is now given.

By writing $\mathbf{A}_{50}$ (5.2) in the following block matrix form in terms of URM3 vectors $\mathbf{X}_{3+}$, $\mathbf{X}^{3-}$ and unity root matrix $\mathbf{A}_{30}(t_3)$, all reproduced below from Appendix (A),

[27] Each dimensional extension of URMT, i.e. URM n to URM $(n+1)$, adds another zero eigenvalue, starting with one zero eigenvalue for URM3, i.e. $\lambda = \pm C, 0$, two for URM4, $\lambda = \pm C, 0, 0$, and three for URM5, $\lambda = \pm C, 0, 0, 0$ etc.

(5.10) $\mathbf{A}_{50} = \begin{pmatrix} 0 & 0 & -t_5\mathbf{X}^{3-} \\ 0 & 0 & -t_4\mathbf{X}^{3-} \\ t_5\mathbf{X}_{3+} & t_4\mathbf{X}_{3+} & \mathbf{A}_{30}(t_3) \end{pmatrix}$

(A1e) $\mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, (A10c) $\mathbf{X}^{3-} = \begin{pmatrix} x & y & -z \end{pmatrix}$, (footnote [28])

(A2b) $\mathbf{A}_{30}(t_3) = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix}$, (footnote [29]),

then $\mathbf{A}_{50}$ can be seen to be decomposed into three, time-dependent matrix components $\mathbf{A}_{53}$, $\mathbf{\Delta}_{54}$ and $\mathbf{\Delta}_{55}$, i.e.

(5.11) $\mathbf{A}_{50} = \mathbf{A}_{53}(t_3) - t_4\mathbf{\Delta}_{54} - t_5\mathbf{\Delta}_{55}$. (footnote [30]),

where $\mathbf{A}_{53}$, $\mathbf{\Delta}_{54}$ and $\mathbf{\Delta}_{55}$ are defined as follows, and $\mathbf{0}_{33}$ is defined as a 3×3 matrix of zeros,

(5.12) $\mathbf{A}_{53} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{A}_{30}(t_3) \end{pmatrix}$,

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[28] The vector $\mathbf{X}^{3-}$ is the 'reciprocal' of $\mathbf{X}_{3+}$ and related via the URM3 matrix operator $\mathbf{T}^3$ ($\sim\mathbf{T}_3$) (H14), as in $\mathbf{X}^{3-} = \left(\mathbf{T}^3\mathbf{X}_{3+}\right)^T$ and, conversely, $\mathbf{X}_{3+} = \left(\mathbf{T}_3\mathbf{X}^{3-}\right)^T$. See Appendix (E) for more information on reciprocal eigenvectors and the $\mathbf{T}$ operator.

[29] Matrix $\mathbf{A}_{30}(t_3)$ is also a function of URM3 evolutionary parameter t_3 since the dynamical variables P, Q, R are functions of t_3, see (A4) and (A5). It is defined in [1],#1 as $\mathbf{A}_{30}(t_3) = \mathbf{A}'_{30} - t_3\mathbf{\Delta}_3^P$, where the primed superscript denotes an initial value as in $\mathbf{A}'_{30} = \mathbf{A}_{30}(t_3 = 0)$, and $\mathbf{\Delta}_3^P$ is defined as $\mathbf{\Delta}^P$ (or $\mathbf{A}_+$) in [1], [3]. Matrix $\mathbf{\Delta}_3^P$ ($\sim\mathbf{\Delta}^P$) is an annihilator, like $\mathbf{\Delta}_{55}$ and $\mathbf{\Delta}_{54}$ (5.14), because it has the property $\mathbf{\Delta}_3^P\mathbf{X}_{3+} = 0$. This decomposition of $\mathbf{A}_{30}(t_3)$ into $\mathbf{A}'_{30}$ and $\mathbf{\Delta}_3^P$ is not required in this paper and provided as background information - it does give some insight into URM3's variational nature and the origin of parameter t_3.

[30] The subscript '54' in $\mathbf{\Delta}_{54}$ denotes the 5×5 matrix for coefficient t_4. Likewise, the subscript '55' in $\mathbf{\Delta}_{55}$ denotes the 5×5 matrix for coefficient t_5.

(5.13) $\mathbf{\Delta}_{54} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{X}^{3-} \\ 0 & -\mathbf{X}_{3+} & \mathbf{0}_{33} \end{pmatrix}$, $\mathbf{\Delta}_{55} = \begin{pmatrix} 0 & 0 & \mathbf{X}^{3-} \\ 0 & 0 & 0 \\ -\mathbf{X}_{3+} & 0 & \mathbf{0}_{33} \end{pmatrix}$, (footnote [31])

The matrices $\mathbf{\Delta}_{55}$ and $\mathbf{\Delta}_{54}$ are known as variational 'delta' matrices in URMT and they have the following annihilator property

(5.14) $\mathbf{\Delta}_{55}\mathbf{X}_{5+} = 0$ and $\mathbf{\Delta}_{54}\mathbf{X}_{5+} = 0$ (footnote [32])

The annihilation property works because the $\mathbf{X}^{3-}$ vector (A10c), embedded in the first and second rows of $\mathbf{\Delta}_{55}$ and $\mathbf{\Delta}_{54}$ respectively, is orthogonal to the $\mathbf{X}_{3+}$ vector (A1e), embedded in the $\mathbf{X}_{5+}$ vector (5.2), i.e.

(5.15) $\mathbf{X}^{3-}\mathbf{X}_{3+} = x^2 + y^2 - z^2 = 0$ Appendix (F1), Pythagoras, orthogonality

The first and second row of matrix product $\mathbf{A}_{50}\mathbf{X}_{5+}$ (5.3) is simply equivalent to the inner product $\mathbf{X}^{3-} \cdot \mathbf{X}_{3+}$ (5.15), which is just the Pythagoras equation, and therefore zero.

Using this annihilator property, the eigenvector equation $\mathbf{A}_{50}\mathbf{X}_{5+}$ (5.3) becomes

(5.16) $\mathbf{A}_{50}\mathbf{X}_{5+} = \mathbf{A}_{53}\mathbf{X}_{5+}$

Writing $\mathbf{X}_{5+}$ (5.2) in block matrix form in terms of $\mathbf{X}_{3+}$

(5.17) $\mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix}$,

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[31] Since $\mathbf{X}_{3+}$ and $\mathbf{X}^{3-}$ are defined purely in terms of coordinates x, y, z, and are completely invariant to variations in t_3, (or t_4 and t_5 for that matter), then the matrices $\mathbf{\Delta}_{55}$ and $\mathbf{\Delta}_{54}$, which comprise $\mathbf{X}_{3+}$ and $\mathbf{X}^{3-}$, are also static, i.e. not a function of time.

[32] The $\mathbf{X}_{3+}$ vector embedded in the first column of $\mathbf{\Delta}_{55}$ and the second column of $\mathbf{\Delta}_{54}$, is seemingly useless since it only multiplies the first two, zero elements of $\mathbf{X}_{5+}$. Whilst it is intentional to have no effect, i.e. remain invariant, it raises the question as to why not use any three arbitrary elements and not, specifically, the $\mathbf{X}_{3+}$ vector? The answer is simple: the $-\mathbf{X}_{3+}$ in the column is the negative conjugate of $\mathbf{X}^{3-}$ in the row and, ultimately, it means dynamical variables H, N, J and their conjugates $\overline{H}, \overline{N}, \overline{J}$ satisfy the Pythagoras conditions (4.2a). This is a must and, since $\mathbf{X}^{3-}$ cannot be chosen arbitrarily (it must satisfy orthogonality (5.15)), it forces the two columns in $\mathbf{\Delta}_{55}$ and $\mathbf{\Delta}_{54}$ to embed $-\mathbf{X}_{3+}$, and not any just any arbitrary vector.

then, from the definition of $\mathbf{A}_{53}$ (5.12) in terms of $\mathbf{A}_{30}(t_3)$ (A2b), the product $\mathbf{A}_{53}\mathbf{X}_{5+}$ in the eigenvector equation (5.16) is effectively the same as $\mathbf{A}_{30}\mathbf{X}_{3+}$ (disregarding dimensionality). Furthermore, since $\mathbf{A}_{30}\mathbf{X}_{3+} = C\mathbf{X}_{3+}$ by its URM3 eigenvector definition (A1f), the original URM5 eigenvector equation (5.3) is restored, i.e. $\mathbf{A}_{50}\mathbf{X}_{5+} = C\mathbf{X}_{5+}$.

The important point here is that the eigenvector equation (5.3) is invariant to arbitrary variations $-t_5\mathbf{\Delta}_{55}$ and $-t_4\mathbf{\Delta}_{54}$ (5.11). The eigenvector equation holds in URM5 just as it does in URM3 (and also URM4 or URMn in general), invariant to any arbitrary variations $-t_5\mathbf{\Delta}_{55}$ and $-t_4\mathbf{\Delta}_{54}$ in matrix $\mathbf{A}_{50}$ (5.10). This might well seem pointless since nothing has been achieved. Which, in a sense, is the whole point of invariance transformations - to do nothing. But, and it's a big but, the transformations do not leave the other four eigenvectors $\mathbf{X}_{5-}$, $\mathbf{X}_{50A}$, $\mathbf{X}_{50B}$, $\mathbf{X}_{50C}$ (for eigenvalues, $\lambda = -C,0,0,0$) invariant, on the contrary, they will change according to the values of t_4 and t_5. Consequently, it is these latter four vectors that generate an evolving eigenvector space in URM5.

If anything, a fair criticism would be that $\mathbf{X}_{5+}$ (5.17) is nothing more than $\mathbf{X}_{3+}$ with a couple of zeros added to the front to extend it from three to five dimensions. However, not all eigenvectors are quite so simple - two of the eigenvectors, $\mathbf{X}_{50B}$ (11.1d) and $\mathbf{X}_{50C}$ (11.1e), have four non-zero elements, and $\mathbf{X}_{5-}$ (11.1b) has a full five, non-zero elements, making it a non-trivial (I7), five-dimensional vector. Although it may seem that such vectors are, therefore, only parameterised in terms of the two variational parameters t_4 and t_5, the solutions themselves are expressed in terms of the URM3 eigenvectors, which are fully parameterised by three parameters, t_3, k and l. Hence they are 5D vectors with a 5D parameterisation (t_3, t_4, t_5, k and l). This has the caveat that not all eigenvectors utilise the full parameterisation. For example, $\mathbf{X}_{50A}$ (11.1c) will be seen to be simply an embedding of the URM3 vector $\mathbf{X}_{30}$, so it is actually only parameterised by the three URM3 parameters, t_3, k and l. Nevertheless, in general, all five parameters are used in the complete solution for URM5 (9.6).

This completes the justification for the form of $\mathbf{A}_{50}$.

Before proceeding to obtain all five eigenvectors from $\mathbf{A}_{50}$, and thereafter analysing the compactification behaviour of URM5, a formal definition of what is meant by compactification, in the context of URMT, has to be given.

6 The Compactification Ratio

Since the aim of the paper is to show compactification occurs over long evolutionary timescales, in one or more excess dimensions, a quantitative measurement of the relative size of a dimension j, with respect to the first three dimensions of URM3, is required. Such a measurement is termed a compactification ratio χ_j, defined further below, in terms of the 'magnitude' of a dimension, which is defined next.

(6.1a) Definition: The **magnitude** (or size), symbol $|\mathbf{X}|_j$, of a particular, excess jth dimension, $j \geq 4$, is a measure (usually an approximation) of the dominant jth coordinate in that dimension. For advance information, this measure is invariably approximated as the time-scaled multiple of the eigenvalue, i.e. $|2t_jC| = 2t_jC$, where $t_j \geq 0$ by convention (7.2) and $C \geq 1$ by definition (3.5).

(6.1b) The magnitude of the first three dimensions, symbol $|\mathbf{X}|_3$, is a measure of the size of the URM3, 3D subspace of the full n-dimensional, eigenvector space. This measure is invariably approximated from just the dominant $\mathbf{X}_{n-}$ vector, and then only using the quadratic term $|t_j^2\mathbf{X}_{3+}|$, $j \geq 4$. That $j \neq 3$ here is intentional, see footnote 33.

(6.2) Definition: The **compactification ratio** of dimension j, denoted by χ_j, is the ratio of the magnitude $|\mathbf{X}|_j$ (6.1a) of the jth dimension to the magnitude $|\mathbf{X}|_3$ (6.1b) of the first three dimensions (URM3), i.e.

(6.3) $$\chi_j = \frac{|\mathbf{X}|_j}{|\mathbf{X}|_3}.$$

With a compactification ratio χ_j defined, then showing compactification occurs over evolutionary timescales translates to showing the ratio χ_j decreases to zero as the jth dimension's evolutionary time t_j increases without bound, i.e.

(6.4) $$\lim_{t_j \to \infty} \chi_j = 0.$$

Since the above definitions for magnitudes $|\mathbf{X}|_j$ and $|\mathbf{X}|_3$ allude to the fact that they are approximated as follows:

(6.5) $$|\mathbf{X}|_j \approx 2t_jC$$

(6.6) $$|\mathbf{X}|_3 \approx |t_j^2\mathbf{X}_{3+}|$$

then the compactification ratio χ_j (6.3), of dimension j, is approximated by

(6.7) $$\chi_j \approx \frac{2C}{t_j|\mathbf{X}_{3+}|},\ t_j \neq 0,\ \text{use (6.3) when } t_j = 0.$$

From this approximation it is seen that χ_j is inversely proportional to time t_j, hence the limit (6.4) is satisfied. Moving on to specifics, the calculation of the compactification ratio, and its behaviour for the URMT eigenvector solutions, is now the main focus of the paper with regard to demonstrating compactification in URMT.

7 Eigenvector Solutions

Equipped with a URM5 unity root matrix $\mathbf{A}_{50}$ (5.10), and a definition for eigenvector $\mathbf{X}_{5+}$ (5.17), the other four eigenvectors $\mathbf{X}_{5-}$, $\mathbf{X}_{50A}$, $\mathbf{X}_{50B}$, $\mathbf{X}_{50C}$, (11.1b) to (11.1e), are determined using what is termed the 'Residual Matrix Method' in URMT. Because this method is outlined in Appendix (C), and fully explained in [1],#2 and [2], the eigenvector solution is quoted further below, without explanation.

Before proceeding to examine the URM5 solution (11.1), it is preferable to study the URM4 solution (8.1) first, which is given in terms of the URM3 eigenvector solution, detailed in Appendices (A) and (B). The reasoning behind this is that URM4 is, of course, the first dimensional extension to URM3 and the arguments on compactification, for URM5 and beyond, are easily established with URM4. It is actually very easy to obtain the URM4 solution (8.1) from the URM5 solution (11.1) by setting the URM5 evolutionary parameter t_5 to zero. It could also be obtained algebraically using $\mathbf{A}_{50}$ (5.10) with $t_5 = 0$, and obtaining the URM4 eigenvectors $\mathbf{X}_{4-}$ (8.1b), $\mathbf{X}_{40A}$ (8.1c) and $\mathbf{X}_{40B}$ (8.1d) from scratch, using the residual method with this cut-down, 4×4 variant '$\mathbf{A}_{40}$' of $\mathbf{A}_{50}(t_5 = 0)$. The $\mathbf{X}_{4+}$ eigenvector is trivially obtained from $\mathbf{X}_{5+}$ (5.17) by eliminating its first, zero element and retaining the remaining four, i.e. $(0, x, y, z)^T \sim \begin{pmatrix} 0 & \mathbf{X}_{3+} \end{pmatrix}^T$.

Three sets of eigenvector solution are given as follows:

(7.1)

1. URM4 in terms of URM3
2. URM5 in terms of URM3
3. URMn in terms of URM3, $n \geq 4$

The URM3 eigenvector solution[33] is given in Appendices (A) and (B).

All solutions are given in block matrix form. Keep in mind all standard (lower subscript) URM3 vectors, e.g. $\mathbf{X}_{3+}$, are 3×1 column vectors, and their reciprocal forms[34] (raised subscript), e.g. $\mathbf{X}^{3-}$, are 1×3 row vectors. Likewise, for URM4, standard forms of eigenvectors such as $\mathbf{X}_{4+}$ are 4×1 column vectors, and their reciprocal forms, e.g. $\mathbf{X}^{4-}$, are

[33] The evolutionary forms of the URM3 eigenvectors, for the solution sets (7.1), have not been expanded in full as functions of parameter t_3, since this isn't particularly necessary for the analysis of compactification of the excess dimensions, and only the behaviour for large evolutionary times, $t_4 >> 0$ and/or $t_5 >> 0$, is required. See Appendix (B) for the URM3 eigenvector equations. The evolutionary behaviour in URM3 is also fully documented in [1],#3, with an overview in [3].

[34] The reciprocal vectors are also often referred to as dual vectors in the literature since they form the basis set, dual to the standard eigenvector basis. However, URMT has a 'dual' formulation, which is not quite the same thing, i.e. it doesn't mean the formulation of URMT in terms of a dual basis. On the contrary, in [1],#5, the dual of $\mathbf{X}_{3+}$ is $\mathbf{X}_{3-}$ and vice-versa, with $\mathbf{X}_{30}$ defined as self-dual, none of which reference the reciprocal vectors. In fact, URMT can be formulated in standard form using $\mathbf{X}_{3+}$, or dual form using $\mathbf{X}_{3-}$, but not both simultaneously - this is URMT's form of duality.

1×4 row vectors. Generally, however, discussion will only refer to the standard vector forms, and not their reciprocals, since every reciprocal vector can be obtained from its standard form using the $\mathbf{T}$ operator relations; see Appendix (E). Identical comments are assumed to apply to the reciprocal forms, as for the standard forms, except where specifically highlighted otherwise.

In all discussion, the first element of the vector is always the nth dimension in URMn, and the remaining $(n-1)$ elements represent the $(n-1)$ dimensions in URM$(n-1)$. The last three elements are always dimensions one to three, i.e. URM3, and referred to as 'the first three dimensions'.

To keep things simple, it will be assumed that all evolution proceeds in the forward, positive direction, i.e.

(7.2) $\quad t_j \geq 0\,,\ \ j = 3 \ldots n$

However, this is convention only, none of the work specifically requires such an assumption, and t_j can be positive or negative. Remember, t_j is a variational parameter and it certainly could be positive, negative, proceeding forward or backward. Nevertheless, using the standard physical interpretation, Section (2), it always has physical units of time.

8 URM4 Eigenvector Evolution Equations

The eigenvector evolution equations for the URM4 eigenvectors, in terms of the URM3 eigenvectors, are as follows:

(8.1)

(8.1a) $\mathbf{X}_{4+} = \begin{pmatrix} 0 \\ \mathbf{X}_{3+} \end{pmatrix}$

(8.1b) $\mathbf{X}_{4-} = -t_4^2 \begin{pmatrix} 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_4 \begin{pmatrix} C \\ \mathbf{0}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{X}_{3-} \end{pmatrix}$

(8.1c) $\mathbf{X}_{40A} = \begin{pmatrix} 0 \\ \mathbf{X}_{30} \end{pmatrix}$

(8.1d) $\mathbf{X}_{40B} = -t_4 \begin{pmatrix} 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} C \\ \mathbf{0}_3 \end{pmatrix}$

Before proceeding with a more detailed analysis in Section (9), the following observations of this solution are made:

(8.2a) Regardless of the size, relative or absolute, of the fourth dimension's evolutionary time t_4, the only vectors that contribute anything to the size of the fourth dimension are $\mathbf{X}_{4-}$ and $\mathbf{X}_{40B}$ since they have a non-zero component, eigenvalue C, as their first element.

(8.2b) In fact, looking ahead to the general, n-dimensional solution, Section (13), C is the only quantity, other than evolutionary time, that is present in each excess dimension, and then it only appears in the linear product term $t_j C$, $j = 4 \ldots n$, or the constant term, as itself, C. Note too that C does not explicitly appear in any of the three URM3 eigenvectors; see (A8). The contribution of the jth dimension at the initial stage of evolution, i.e. $t_j = 0$, is thus governed by the magnitude of C compared to the magnitude of the URM3 eigenvectors. Given C is related to the total energy, $E = C^2$ (2.1), it means that at $t_j = 0$ a comparatively large value for C would make for a sizeable excess dimension with a lot of energy in it, C being suitably chosen as an initial condition.

(8.3) For any sufficiently large[35] evolutionary time t_4, the URM3 vector $\mathbf{X}_{3+}$ dominates the entire solution. See, for example, $\mathbf{X}_{4-}$ (8.1b) with a quadratic '$-t_4^2$' term. Since $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$

[35] The caveat 'sufficiently large evolutionary time' appears repeatedly throughout. In general it means any time t large enough such that the approximation under discussion is valid. In actuality it means the magnitude of the quadratic term in t_j^2, in the $\mathbf{X}_{n-}$ eigenvector, for excess dimension j, $j = 4 \ldots n$, dominates all other terms in all eigenvectors. It is discussed again in Section (10).

are functions of the evolutionary time t_3, Appendix (B), they also grow with t_3. However, for a large URM3 evolutionary time t_3, regardless of time t_4, URM3 itself converges (flattens) to also align with $\mathbf{X}_{3+}$ and, ultimately, all evolution tends to align with $\mathbf{X}_{3+}$. Nevertheless, that any excess dimension is dominated by one or more of the URM3 vectors, $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$, only serves to bolster arguments that the compactification reduces from the higher, excess dimensions to those of URM3. Suffice to note, it is not the relative size of URM3 vectors that matters with regard to compactification of excess dimensions, but the size of the excess dimensions relative to those spanned by the URM3 vectors.

9 Analysis of URM4 Compactification

Looking at the URM4 eigenvector solution (8.1) in more detail, the only two vectors in URM4 that contribute to the fourth dimension are $\mathbf{X}_{4-}$ and $\mathbf{X}_{40}$, i.e. they have a non-zero first element. In vector $\mathbf{X}_{4-}$, the size of the fourth dimension (first element) is controlled by the linear term $2t_4C$, and in vector $\mathbf{X}_{40}$, the size of the fourth dimension is controlled by the constant term C. There are a few sensible ways to combine these two sizes, e.g. root sum squares or summation of the magnitude of the individual components, i.e.

(9.1) $\quad |\mathbf{X}|_4 = \sqrt{(2t_4C)^2 + C^2}$ or $|\mathbf{X}|_4 = |2t_4C| + |C|$,

but, given the analysis is primarily interested in large evolutionary times, i.e. $t_4 >> 0$, it is clear that only the $2t_4C$ term from $\mathbf{X}_{4-}$ will dominate, i.e.

(9.2) $\quad 2t_4C >> C$ for $t_4 >> 0$.

Therefore, the magnitude of the fourth dimension is simply approximated as the magnitude of the time-dependent component, i.e.

(9.3) $\quad |\mathbf{X}|_4 \approx 2t_4C,\ t_4 >> 0$.

From here onward, to avoid repetition, the following two points are assumed throughout this section, and stated here

(9.4a) All calculations of $|\mathbf{X}|_3$ are restricted to their URM3 components only (last three elements).

(9.4b) All approximations for $|\mathbf{X}|_3$ are assumed valid for sufficiently large t_4 (footnote [36]), with some justification given in Section (10).

[36] Note that there is a subtle distinction here between a large time t_4, as in much greater than zero ($t_4 >> 0$), and a 'sufficiently large' t_4, such that any approximation is actually valid. In the first case (9.3), of sizing the fourth dimensional component, the $t_4 >> 0$ criterion is sufficient given (9.2). In the second case, of sizing the other three dimensions, having $t_4 >> 0$ might not, by itself, be sufficient to justify the approximation, with a more exact definition required. This topic is considered again in Section (10).

The magnitude of the first three dimensions, denoted by $\left|\mathbf{X}\right|_3$, might seem a much messier affair because there are now three URM3 vectors $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ embedded within the URM3 dimensions (last three elements) of URM4. Looking at the URM4 eigenvectors (8.1), the URM3 components for each vector are

(9.5)

(9.5a) $\mathbf{X}_{4+} = \mathbf{X}_{3+}$, URM3 components only

(9.5b) $\mathbf{X}_{4-} = -t_4^2 \mathbf{X}_{3+} + \mathbf{X}_{3-}$, ditto

(9.5c) $\mathbf{X}_{40A} = \mathbf{X}_{30}$, ditto

(9.5d) $\mathbf{X}_{40B} = -t_4 \mathbf{X}_{3+}$, ditto

A measure of $\left|\mathbf{X}\right|_3$ can be obtained by combining the magnitudes of these above components and, again, a root sum squares or sum of individual vector magnitudes are the two common methods of combination[37]:

(9.6)

(9.6a) $\left|\mathbf{X}\right|_3 = \sqrt{\left|\mathbf{X}_{4+}\right|^2 + \left|\mathbf{X}_{40A}\right|^2 + \left|\mathbf{X}_{40B}\right|^2 + \left|\mathbf{X}_{4-}\right|^2}$, root sum of squares.

(9.6b) $\left|\mathbf{X}\right|_3 = \left|\mathbf{X}_{4+}\right| + \left|\mathbf{X}_{40A}\right| + \left|\mathbf{X}_{40B}\right| + \left|\mathbf{X}_{4-}\right|$, sum of magnitudes

Of the four vectors (9.5) in these expressions, only $\mathbf{X}_{4-}$ (9.5b) is dominant because it is the only vector with a quadratic term in t_4. Therefore, $\left|\mathbf{X}\right|_3$ can be approximated by $\left|\mathbf{X}_{4-}\right|$ for some sufficiently large time t_4, i.e.

(9.7) $\left|\mathbf{X}\right|_3 \approx \left|\mathbf{X}_{4-}\right|$.

Furthermore, (9.5b) shows that $\mathbf{X}_{4-}$ is dominated by the term $-t_4^2 \mathbf{X}_{3+}$, when assuming the following, which is basically the criteria of a 'sufficiently large time t_4,

(9.8) $\left|t_4^2 \mathbf{X}_{3+}\right| >> \left|\mathbf{X}_{3-}\right|$.

[37] It should be noted that all URMn vector spaces are, generally, highly oblique, i.e. the eigenvectors, as a basis, are far from orthogonal to each other, and neither are they are of unit magnitude. Therefore, any such measures (root sum squares, etc.) are relatively basic estimates, but considered acceptable if consistently applied.

Under this assumption, $\mathbf{X}_{4-}$ is approximated by $-t_4^2\mathbf{X}_{3+}$, formalised as

(9.9) $\left|\mathbf{X}_{4-}\right| \approx \left|t_4^2\mathbf{X}_{3+}\right|$.

(9.10)
However, although not shown, $\mathbf{X}_{3-}$ also evolves with URM3's evolutionary time t_3, and it too can be approximated, for some sufficiently large t_3, by $\left|t_3^2\mathbf{X}_{3+}\right|$, i.e. $\left|\mathbf{X}_{3-}\right| \approx \left|t_3^2\mathbf{X}_{3+}\right|$. Thus, it is easily possible a time t_3 can be found such that the magnitudes $\left|\mathbf{X}_{3-}\right|$ and $\left|t_4^2\mathbf{X}_{3+}\right|$ are comparable, i.e. $\left|t_4^2\mathbf{X}_{3+}\right| \approx \left|\mathbf{X}_{3-}\right| \approx \left|t_3^2\mathbf{X}_{3+}\right|$. It might then be better to use a combined estimate such as $\left|\left(t_4^2 + t_3^2\right)\mathbf{X}_{3+}\right|$ for $\left|\mathbf{X}_{4-}\right|$. In fact, the first, simpler approximation $\left|t_4^2\mathbf{X}_{3+}\right|$ will be used, ignoring t_3 completely, by virtue of the explanation given next.

(9.11)
Fortunately, these aforementioned concerns about t_3, and the comparative size of its evolving vectors, are all irrelevant for the following reason: given the definition of χ_j has the URM3 magnitude $\left|\mathbf{X}\right|_3$ in its denominator, a smaller estimate for $\left|\mathbf{X}\right|_3$ will give a larger ratio χ_j, i.e. a more pessimistic measure of compactification. If a pessimistic measure of χ_j can be shown to converge to zero, for large evolutionary times, then it will also converge to zero quicker, i.e. for smaller evolutionary times, when the true magnitude of $\left|\mathbf{X}\right|_3$ is greater than that used in the calculation.

Since $\left|\mathbf{X}\right|_3$ is dominated by $\left|\mathbf{X}_{4-}\right|$ due to the domination of the term $-t_4^2\mathbf{X}_{3+}$, and disregarding any URM3 contribution due to t_3 for reasons given above (an increasing t_3 only grows the relative size of URM3, and betters the compactification), then the approximation $\left|-t_4^2\mathbf{X}_{3+}\right|$ will be used as measure $\left|\mathbf{X}\right|_3$ of the magnitude of the last three dimensions of URM4, i.e.

(9.12) $\left|\mathbf{X}\right|_3 \approx t_4^2\left|\mathbf{X}_{3+}\right|$.

Having established approximations for $\left|\mathbf{X}\right|_4$ (9.3) and $\left|\mathbf{X}\right|_3$ (9.12), the URM4 compactification ratio χ_4 can now be calculated.

The URM4 Compactification Ratio χ_4

Substituting the approximation for $|\mathbf{X}|_4$ (9.3), and $|\mathbf{X}|_3$ (9.12) into (6.3), for $j = 4$, the compactification ratio χ_4 of the fourth dimension in URM4, is approximated as follows; see also Section (10) which shows this approximation improves as t_4 increases.

(9.13) $\chi_4 \approx \dfrac{2C}{t_4|\mathbf{X}_{3+}|}$, for sufficiently large t_4,

Note that $\chi_4 \geq 0$ since $C > 0$ by definition (3.5), and $t_4 > 0$ by convention[38] (7.2). This convention is strictly $t_4 \geq 0$, but the ratio is only calculated for $t_4 > 0$ for the obvious reason to avoid a zero divisor in (9.13). A different compactification ratio for t_4, at time zero, could be calculated but it is rather pointless since the focus is on large evolutionary times.

The vector magnitude $|\mathbf{X}_{3+}|$ (footnote[39]) is constant with respect to time because the vector $\mathbf{X}_{3+}$ is static (I11), i.e. it has no dependence on any evolutionary parameter, notably t_3; see also footnote 7. So too is eigenvalue C also a constant and, additionally, an initial condition. The ratio χ_4 is therefore just inversely proportional to the time t_4, and so tends to zero as t_4 tends to infinity, i.e.

(9.14) $\lim\limits_{t_4 \to \infty} \chi_4 = 0$.

To conclude then, in the four-dimensional vector space of URM4, the excess, fourth dimension is seen to contract as the evolutionary time t_4, for that dimension, grows ever larger, eventually appearing to have zero size as t_4 grows infinite. Hence, under assumption (9.4b), the four-dimensional vector space compactifies to that of the eigenvector space of URM3, as evolution progresses. Specifically, all 4D eigenvectors align with the single, static URM3 eigenvector $\mathbf{X}_{3+}$, which occupies a 2D subspace (footnote 7) of URM3; hence URM4 compactifies to appear two-dimensional.

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[38] The convention is actually $t_4 \geq 0$ but the ratio is only calculated for $t_4 > 0$ for obvious reasons in (9.13). A different compactification ratio for t_4 at time zero could be calculated but, since the focus is on large evolutionary times, it is rather pointless.

[39] By the definition (A1e) of $\mathbf{X}_{3+}$ in terms of acceleration coordinates x, y, z, the magnitude $|\mathbf{X}_{3+}|$ is actually $|\mathbf{X}_{3+}| = \sqrt{2}|z|$. By the Pythagorean relation (F1) between x, y, z, $|z|$ is always greater than zero, see (A3c) and (A3d), and increases with increasing values for parameters k and l, hence only increasing compactification by decreasing the compactification ratio.

As regards URM4, it remains to define what is meant by a 'sufficiently large evolutionary time', and justify the assumption (9.4b) made. This follows next, and is applicable to any excess dimension j, $j = 4 \ldots n$.

Following this, the same compactification analysis is performed on the 5D solution, which is seen to have evolutionary terms in both t_4 and t_5.

10 A Sufficiently Large Evolutionary Time

Until now, the term 'sufficiently large evolutionary time' has been considered as any evolutionary time t, large enough such that the approximation under discussion is valid. Now that a specific approximation for the size $|\mathbf{X}|_3$ of the URM3 dimensions, embedded within URM4, has been given, i.e. $\left|t_4^2\mathbf{X}_{3+}\right|$ (9.12), the term 'sufficiently large evolutionary time' can be made more definitive.

Although only URM4 has been analysed, this section will generalise to the jth dimension, $j = 4 \ldots n$, for URM n, which basically just means replacing every subscript of '4' by 'j', as regards results obtained in the previous section.

Firstly, the same approximations used in URM4 are now generalised for an arbitrary dimension j, $j = 4 \ldots n$, as follows:

(10.1)
(10.1a) $|\mathbf{X}|_3 \approx \left|\mathbf{X}_{j-}\right|$, see (9.7) for URM4
(10.1b) $\left|\mathbf{X}_{j-}\right| \approx \left|t_j^2\mathbf{X}_{3+}\right|$, see (9.9) for URM4.

The two above approximations are combined to give

(10.1c) $|\mathbf{X}|_3 \approx \left|t_j^2\mathbf{X}_{3+}\right|$, see (9.12) for URM4.

Readers are also referred to the general solution in Section (13) to see these approximations.

A measure of the relative error ε in the approximation (10.1c) of $|\mathbf{X}|_3$ by $\left|t_j^2\mathbf{X}_{3+}\right|$, at any time t_j, is given by

(10.2) $\varepsilon = \dfrac{\left||\mathbf{X}|_3 - \left|t_j^2\mathbf{X}_{3+}\right|\right|}{|\mathbf{X}|_3}$, estimate of relative error in approximation at time t_j.

(10.3)
It is at this stage that a potential problem appears. Whilst the approximation (10.1c) $\left|t_j^2\mathbf{X}_{3+}\right|$ is acceptable when calculating the compactification ratio χ_j (χ_4) (9.13), using just time t_j (t_4) (for reasons outlined in (9.10) and (9.11)), it is not so good when calculating the relative error ε in (10.2). The idea behind the calculation of ε, as seen further below in this section, is that it removes the quadratic term t_j^2 from the numerator in (10.2), leaving only linear terms in t_j. However, by ignoring all other evolutionary times t_i, where $i \neq j$, $i, j = 3 \ldots n$, from approximation (10.1c), and using just t_j, leaves quadratic terms in t_i which can be as large as t_j, if not larger. In other words, the error ε is not small in these circumstances, i.e. when

ignoring t_i, and it will not always converge to zero as t_j grows, even though the compactification ratio will converge, as per (9.14).

There are three methods to overcome this:

(10.3a) Replace the crude approximation $\left|t_j^2 \mathbf{X}_{3+}\right|$ by a better approximation $\left|T_n^2 \mathbf{X}_{3+}\right|$, where $T_n^2 = \sum_{k=3}^{n} t_k^2$. This is alluded to in point (9.10) where it is suggested that $\left|\left(t_4^2 + t_3^2\right)\mathbf{X}_{3+}\right|$ be used instead of $\left|t_4^2 \mathbf{X}_{3+}\right|$.

(10.3b) Assert $t_j >> t_i$, $i \neq j$, i.e. make the jth evolutionary time much greater than all others t_i.

(10.3c) Set all $t_i = 0$, where $i \neq j$, $i, j = 3 \ldots n$

The first (10.3a) seems a good, obvious choice since the original approximation is very crude and, in reality, this better approximation, using T_n^2, should always be used for both an accurate calculation of the compactification ratio and in any error analysis. Nevertheless, it will not be used here solely because it makes the analysis clumsy and, most importantly, it isn't really necessary because the third option, (10.3c) below, circumvents the problem.

The second choice (10.3b) will do the job, i.e. make the approximation (10.1c) reasonable, but is disliked because it means the evolution times can never be comparable, i.e. it becomes a condition that the jth evolutionary time t_j is always much greater than every other, ith time t_i. Since the evolutionary times may well all be identical, this solution is not acceptable except when all other times t_i are zero. This then is the third solution (10.3c), discussed next.

The third choice (10.3c) is the preferred option because it makes the analysis simple and will make the approximation (10.1c) valid, even if it is an artificial condition. Although artificial in that all evolutionary times are zero, other than t_j, the computation of χ_j remains unchanged and valid. As noted for URM4, points (9.10) and (9.11), ignoring non-zero t_i will give a worst-case estimation of χ_j, and any non-zero times t_i will only make χ_j better (smaller), i.e. faster compactification.

Lastly on this issue, if true accuracy is required, it is a simple matter to revert to method (10.3a), i.e. replace time t_j^2 with the combined, quadratic time T_n^2 in the approximation of $|\mathbf{X}|_3$ (10.1c), which can then be used to calculate χ_j.

To conclude the above, for the purposes of this section only, the calculations will assume all times t_i, other than t_j, are zero; t_j being both non-zero and likely 'large'

(10.4) $t_j \neq 0$, $t_i = 0$, where $i \neq j$, $i, j = 3 \ldots n$.

Returning then to the calculation of the relative error ε (10.2), under the assumption (10.4), this calculation requires a true (accurate) expression for $|\mathbf{X}|_3$. This was left undecided in the previous section, with one of two options, (9.6a) and (9.6b), available. Given $|\mathbf{X}_{j-}|$ is the dominant term in $|\mathbf{X}|_3$ then, whichever one of the two is chosen, they both approximate to (10.1b), (10.1c). The best form for analysis is the sum of magnitudes, e.g. (9.6b), and is thus chosen as a measure of $|\mathbf{X}|_3$, now formally defined by

(10.5) $|\mathbf{X}|_3 = \sum_{i=1}^{n} |\mathbf{X}_i|$, for all eigenvectors $\mathbf{X}_i$ in the n-dimensional basis.

To see how the relative error ε behaves with respect to time t_j, it is also useful to define an absolute error ε_- for the approximation (10.1b), calculated as follows,

(10.6) $\varepsilon_- = |\mathbf{X}_{j-}| - |t_j^2 \mathbf{X}_{3+}|$.

Using this, and the sum form (10.5) for $|\mathbf{X}|_3$, then the numerator of ε (10.2) is re-written as

(10.7) $\left\| \mathbf{X}|_3 - |t_j^2 \mathbf{X}_{3+}| \right| = \left(|\mathbf{X}_{n+}| + |\mathbf{X}_{n0}| + \sum_{i=1}^{n-3} |\mathbf{X}_{n0i}| \right) + \varepsilon_-$.

For example, using the URM4 eigenvectors (9.5), and assuming ε_- is small, this becomes

(10.8) $\left\| \mathbf{X}|_3 - |t_4^2 \mathbf{X}_{3+}| \right| \approx \left(|\mathbf{X}_{4+}| + |\mathbf{X}_{40A}| + |\mathbf{X}_{40B}| \right)$, URM4.

Looking at (10.7), and ahead to (13.2b) for $|\mathbf{X}_{j-}|$, then because the quadratic term in t_j^2 has been removed by the subtraction of $|t_j^2 \mathbf{X}_{3+}|$ on the left of (10.7), ε_- is only of linear order in time t_j (t_4), for large t_j, and ignoring t_i (t_3) by assumption (10.4), then

(10.9) $O(\varepsilon_-) = t_j$.

Likewise, from the general solution (13.2), or using the URM4 vectors (9.5) as an example, all vectors in the bracketed term on the right of (10.7) and (10.8) are also only of linear order, since $\mathbf{X}_{j-}$ ($\mathbf{X}_{4-}$) is the only vector with a quadratic factor in t_j (t_4), i.e.

(10.10) $O\left(\left|\mathbf{X}_{n+}\right| + \left|\mathbf{X}_{n0}\right| + \sum_{i=1}^{n-3}\left|\mathbf{X}_{n0i}\right|\right) = t_j$.

Therefore, the entire order of the numerator term (10.7) is linear in t_j.

(10.11) $O\left\|\mathbf{X}\right|_3 - \left|t_j^2\mathbf{X}_{3+}\right\| = t_j$.

Conversely, the denominator $\left|\mathbf{X}\right|_3$ still contains $\left|\mathbf{X}_{j-}\right|$ and remains a quadratic function of t_j, i.e.

(10.12) $O\left|\mathbf{X}\right|_3 = t_j^2$

Therefore, inserting numerator (10.7) and denominator (10.5) into (10.2) shows that the error ε is now inversely proportional to time t_j, i.e.

(10.13) $\varepsilon \propto \frac{1}{t_j}$, under assumption (10.3c), footnote[40]

In other words, choosing $\left|\mathbf{X}\right|_3$ as the form (10.5), and using the quadratic approximation (10.1c), gives an estimate for the approximation error ε, which is inversely proportional to time and, thus, decreases to zero as time increases.

This last result (10.13) is pivotal in defining the term 'sufficiently large...' because, basically, it means that, for any time greater than t_j, the relative error ε will always be less than its value at time t_j, which is formalised next.

Finally then, by choosing a value of the maximum, permissible error ε_j as a pre-condition:

(10.14) ε_j = the maximum, permissible error for ε (10.2), e.g. 0.01 for 1% error,

then a definition for 'sufficiently large evolutionary time' is given as follows:

(10.15) Definition: a **sufficiently large evolutionary time** is considered to be a time t_j, for a specific dimension j, $j = 4 \ldots n$, if, for all times t greater than t_j, the relative error ε (10.2)

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[40] If the assumption (10.3c) is unpalatable, then follow the suggestion in (10.3a), i.e. replace t_j^2 with a better, more accurate time T_n^2 in both the calculation of χ and ε. Doing so will then give the same, inverse-time result (10.13) for ε.

in the estimate for the size $|\mathbf{X}|_3$ of the first three dimensions, i.e. the size of URM3 embedded in URMn, is less than ε_j, i.e.

if $\varepsilon < \varepsilon_j$ for all $t > t_j$ then t_j is 'sufficiently large'

Furthermore, the error decreases with increasing time such that it converges to zero, i.e.

(10.16) $\lim_{t \to \infty} \varepsilon = 0$

Admittedly, this does not give any actual sufficiency time, but merely shows that by approximating the magnitude $|\mathbf{X}|_3$ by selecting the dominant, quadratic, evolutionary terms in t_j, under certain assumptions (10.4b) (which can be rectified - footnote 40), the error in this approximation converges to zero as evolutionary time progresses. The numerical example in Appendix (D) provides some values of ε versus t_j.

11 URM5 Eigenvector Evolution Equations

The eigenvector evolution equations for the URM5 eigenvectors, in terms of the URM3 eigenvectors, are as follows:

(11.1)

(11.1a) $$\mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix}$$

(11.1b) $$\mathbf{X}_{5-} = -(t_5^2 + t_4^2)\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_5\begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + 2t_4\begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3-} \end{pmatrix}$$

(11.1c) $$\mathbf{X}_{50A} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{30} \end{pmatrix}$$

(11.1d) $$\mathbf{X}_{50B} = -t_4\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix}$$

(11.1e) $$\mathbf{X}_{50C} = -t_5\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix}$$

Looking at this solution, there are no mixed t_4, t_5 terms, and it splits nicely into independent terms in t_4 and t_5. The vector $\mathbf{X}_{5-}$, that contains terms in t_4 and t_5, is split into its 5D and 4D components, denoted by $\mathbf{X}_{5-}(t_4)$ and $\mathbf{X}_{5-}(t_5)$, as follows:

(11.2)

(11.2a) $$\mathbf{X}_{5-}(t_4) = -t_4^2\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_4\begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3-} \end{pmatrix}$$

(11.2b) $$\mathbf{X}_{5-}(t_5) = -t_5^2\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_5\begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3-} \end{pmatrix}$$

(11.2c) $$\mathbf{X}_{5-} = \mathbf{X}_{5-}(t_4) + \mathbf{X}_{5-}(t_5)$$

The zero eigenvectors (I12) $\mathbf{X}_{50B}$ and $\mathbf{X}_{50C}$ are also, rather conveniently, already separated into a 4D and 5D, time-dependent form[41], with each uniquely associated to its dimension, i.e. $\mathbf{X}_{40B}$ with the fourth, and $\mathbf{X}_{50C}$ with the fifth, trivially expressed as

(11.2d) $$\mathbf{X}_{50B}(t_4) \equiv \mathbf{X}_{50B}$$

(11.2e) $$\mathbf{X}_{50C}(t_5) \equiv \mathbf{X}_{50C},$$

Not only does the URM5 solution separate into a unique 4D and 5D term, but these terms are identical, disregarding the particular excess dimension. They are 'identical' in so far as the URM4 component contributes a linear term $2t_4C$ in $\mathbf{X}_{5-}(t_4)$ and C in $\mathbf{X}_{50B}(t_4)$, and the URM5 component contributes a term $2t_5C$ in $\mathbf{X}_{5-}(t_5)$ and C in $\mathbf{X}_{50C}(t_5)$, which are identical upon interchange of times t_4 and t_5. Since it is only the magnitude of the contribution in the excess dimension that matters in the analysis of the ratio χ, each dimension can be treated separately and, furthermore, identical findings for one dimension, apply to the other.

Concluding from the previous paragraph, the separability of the URM5 solution into 4D and 5D unique components, and the interchange symmetry of t_4 and t_5 between the two components (as regards calculating the ratio χ), means that each dimension acts independently, with identical behaviour with respect to their individual evolutionary times.

The compactification behaviour of the fifth dimension, isolated from the 4D behaviour, can easily be analysed by equating time t_4 to zero, and thereby nullifying the 4D component, leaving only the 5D solution in terms of t_5. However, as stated, barring the fact that the 5D component affects the fifth dimension, and not the fourth, it is effectively the same solution as that for URM4 (8.1), by virtue of the interchange symmetry between t_4 and t_5. Thus, the same arguments used for URM4 can be applied to URM5 and, most importantly, the expression for χ_5, i.e. the URM5 equivalent of URM4's ratio χ_4 (9.13), is simply written down by interchange of t_4 with t_5, i.e.

(11.3) $$\chi_5 \approx \frac{2C}{t_5|\mathbf{X}_{3+}|}, \text{ for sufficiently large } t_5,$$

As regards URM5, it only remains to examine when both t_4 and t_5 are comparably large, i.e.

(11.4) $$t_5 \approx t_4$$

Unsurprisingly, given the above discussion, if either one of t_4 or t_5 is sufficiently large, then compactification will still occur since the dimensions act independently. If both t_4 and t_5 are

[41] Time-dependent for excess dimensions only, and not $\mathbf{X}_{50A}$ (11.1c), which is simply an embedding of the URM3 vector $\mathbf{X}_{30}$, and a function of URM3's time t_3.

large, it will only serve to increase this compactification process further. However, this process is nicely illustrated with regard to what is really, physically happening, and that is that URM3, from the perspective of the fourth or fifth dimension, appears to expand with a constant acceleration.

12 An Expanding URM3 Vector Space

Whilst the analysis and discussion is focussed on the concept of compactification, i.e. relative shrinkage of dimensions, it is not that any dimension actually shrinks, but rather that the URM3 dimensions appear to expand, and then with a constant acceleration (constant with respect to evolutionary time), that of the URM3 static vector $\mathbf{X}_{3+}$ [42]. Simultaneously, the excess dimensions expand linearly with a constant velocity (eigenvalue C). Because the 3D expansion is a static acceleration, URM3 spatially expands along $\mathbf{X}_{3+}$ (within $\mathbf{X}_{5-}$) quadratically with respect to the evolutionary times t_4 and t_5, as witnessed by the solution for vector $\mathbf{X}_{5-}$ (11.1b); there is also the linear expansion (due to velocity C) in the excess dimensions of both $\mathbf{X}_{5-}$ and the zero vectors $\mathbf{X}_{50B}$ (11.1d) and $\mathbf{X}_{50C}$ (11.1e). But, of course, these linear terms becomes less important relative to the quadratic term, as the evolution progresses, hence the apparent contraction of the excess dimensions relative to URM3.

Looking at the URM5 eigenvector solution (11.1), it is clear, for sufficiently large t_4 and/or t_5, the solution is dominated by the quadratic term in $\mathbf{X}_{5-}$ scaling the vector $\mathbf{X}_{5+}$. This vector $\mathbf{X}_{5+}$ is really just the URM3 vector $\mathbf{X}_{3+}$ embedded in URM5 with a zero fourth and fifth dimensional contribution. As noted earlier, (9.10) and (9.11), ignoring the vector $\mathbf{X}_{3-}$ gives a worst-case ('pessimistic') compactification ratio. Therefore, just concentrating on $\mathbf{X}_{5-}$, it is approximated as follows, for sufficiently large t_4 and/or t_5,

(12.1) $\mathbf{X}_{5-} \approx -(t_5^2 + t_4^2)\mathbf{X}_{5+}$.

Given $\mathbf{X}_{5+}$ is really just a 5D embedding of $\mathbf{X}_{3+}$, the URM3 vector space grows quadratically with respect to either t_4 or t_5, along $\mathbf{X}_{3+}$, and it matters not if t_4 is small relative to t_5 ($t_5 >> t_4$), providing t_5 is sufficiently large. Likewise, in the converse case, $t_4 >> t_5$, there is still quadratic growth in URM3 along $\mathbf{X}_{3+}$. Thus, either the fourth or fifth dimension can act in isolation to increase the size of the URM3 space, by growth in its evolutionary parameter, t_4 or t_5 respectively; both evolutionary times acting together can only increase this growth further. Given all excess dimensions only grow linearly with evolutionary time, the quadratic growth in $\mathbf{X}_{3+}$ will have the desired effect of making all the excess dimensions appear to shrink (compactify) and align along $\mathbf{X}_{3+}$, as also happens in URM3 for large evolutionary periods t_3.

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[42] This is the reason why all the other eigenvectors align with $\mathbf{X}_{3+}$ over times t_j, since they are related to $\mathbf{X}_{3+}$ by calculus relations, e.g. the velocity $\mathbf{X}_{n0j}$ is the integral of acceleration $\mathbf{X}_{n+}$ ($\sim\mathbf{X}_{3+}$), and the position $\mathbf{X}_{n-}$ is the corresponding integral of the velocity (both to within a constant factor), see Appendix (G) for more details.

To summarise, when both evolution times are comparable, and at least one of them is sufficiently large, then the compactification process still occurs. Thus, for any large, fourth or fifth-dimension evolutionary time, the five-dimensional vector space compactifies to that of the eigenvector space of URM3 as evolution progresses, for 'sufficiently large evolutionary times' t_4 or t_5. Specifically, all 5D eigenvectors align with the single, static URM3 eigenvector $\mathbf{X}_{3+}$, which occupies the discrete, 2D, conical subspace of URM3, hence URM5, like URM4, also compactifies to appear two-dimensional within the 3D space of URM3..

The final stage then is to show that this same compactification behaviour arises for any arbitrary, n-dimensional space, which is demonstrated using the general, n-dimensional solution.

13 The General n-dimensional Solution

The general solution for URMn, $n \geq 4$, is actually obtained recursively by calculating the residual matrix for $\mathbf{A}_{n0}$ using an embedded matrix $\mathbf{A}_{(n-1)0}$; Appendix (C) gives an outline. However, the recursive solution is best given in a much simpler, unravelled form in terms of the URM3 vectors. Barring a single, linear Diophantine equation in URM3, see (A4), URMn is a completely solved problem with an analytic solution for the eigenvectors parameterised by all n parameters, i.e. $n-2$ temporal parameters t_j, $j = 3\ldots n$, and two non-temporal parameters k and l (A3d).

With $\mathbf{0}_{33}$ defined as a 3×3 matrix of zeros, and $\mathbf{I}_{n-3}$ as the $(n-3)\times(n-3)$ identity matrix, then a constant, $n \times n$ matrix $\mathbf{M}_n$ is constructed as follows:

(13.1) $$\mathbf{M}_n = C\begin{pmatrix} \mathbf{I}_{n-3} & 0 \\ 0 & \mathbf{0}_{33} \end{pmatrix}$$

The subscript n on $\mathbf{M}_n$ will be dropped from here onward and $\mathbf{M}$ assumed a square $n \times n$ matrix.

The matrix $\mathbf{M}$ has a lead diagonal with all elements equal to eigenvalue C except for the last three diagonal elements, which are zero. These last three zeros are, of course, so that the matrix $\mathbf{M}$ has no URM3, 3D contribution. Equally importantly, $\mathbf{M}$ is a constant matrix and has no time dependence; the ith column[43] of $\mathbf{M}$ is the equivalent of the initial value, zero eigenvector at $t_{j+3} = 0$, denoted by primed vector $\mathbf{X}'_{n0j}$, see further below.

The general solution for the vector $\mathbf{X}_{n+}$ is nothing more than the embedding of the static, URM3 vector $\mathbf{X}_{3+}$ in its last three elements, padded with $n-3$ leading zeros.

(13.2)

(13.2a) $$\mathbf{X}_{n+} = \begin{pmatrix} \mathbf{0}_{n-3} \\ \mathbf{X}_{3+} \end{pmatrix}$$

Denoting the kth element of $\mathbf{X}_{n-}$ by $\mathbf{X}_{n-,k}$, $k = 1\ldots n$, then the general solution for vector $\mathbf{X}_{n-}$ is given by

(13.2b) $$\mathbf{X}_{n-,k} = \begin{pmatrix} \mathbf{0}_{n-3} \\ \mathbf{X}_{3-} \end{pmatrix}_k + \sum_{j=1}^{n-3}\left(-t_{j+3}^2 \begin{pmatrix} \mathbf{0}_{n-3} \\ \mathbf{X}_{3+} \end{pmatrix}_k + 2t_{j+3}\mathbf{M}_{ki} \right),$$

$k = 1\ldots n$, $n \geq 4$, $i = n-(j+2)$, footnote 43

[43] The index i, as in $i = n-(j+2)$, (13.2b), goes from $n-3$ to 1 as j goes from 1 to $n-3$, and works across the first $n-3$ columns of $\mathbf{M}$, which are non-zero, unlike the last three columns.

The first zero vector $\mathbf{X}_{n0A}$ ($\sim\mathbf{X}_{n0j}$, $j=0$) is just an embedding of the URM3 zero vector $\mathbf{X}_{30}$

$$(13.2c)\qquad \mathbf{X}_{n0A}=\begin{pmatrix}\mathbf{0}_{n-3}\\ \mathbf{X}_{30}\end{pmatrix},\ \mathbf{X}_{n0A}\sim\mathbf{X}_{n0j},\ j=0.$$

For all other zero vectors, denoting the kth element of $\mathbf{X}_{n0j}$ by $\mathbf{X}_{n0j,k}$, $k=1\ldots n$, then the kth element of the jth, zero eigenvector $\mathbf{X}_{n0j}$, $j=1\ldots n-3$, is given by

$$(13.2d)\ \mathbf{X}_{n0j,k}=-t_{j+3}\begin{pmatrix}\mathbf{0}_{n-3}\\ \mathbf{X}_{3+}\end{pmatrix}_k+\mathbf{M}_{ki},$$

$$j=1\ldots n-3,\ k=1\ldots n,\ n\geq 4,\ i=n-(j+2).$$

$\mathbf{X}_{n0B}\sim\mathbf{X}_{n01}$ ($j=1$), $\mathbf{X}_{n0C}\sim\mathbf{X}_{n02}$ ($j=2$) etc.

By setting the evolutionary time t_{j+3} to zero, it is seen that the ith column of $\mathbf{M}$, where $i=n-(j+2)$, is the initial, zero vector $\mathbf{X}'_{n0j}$, for the jth dimension, $j=1\ldots n-3$ ($i=n-3\ldots 1$), i.e.

$$(13.3)\quad \mathbf{X}'_{n0j,k}=\mathbf{M}_{ki}\ \text{at}\ t_{j+3}=0,\ j=1\ldots n-3,\ k=1\ldots n,\ n\geq 4,\ i=n-(j+2).$$

e.g. for URM4, $n=4$, $j=1$, $i=1$, $k=1\ldots 4$, $\mathbf{X}'_{401,k}=\mathbf{M}_{k1}$

$$(13.4)\quad \mathbf{M}=\begin{pmatrix}C & 0\\ \mathbf{0}_3 & \mathbf{0}_{33}\end{pmatrix},\ \mathbf{M}_{k1}=\mathbf{X}'_{401}=\begin{pmatrix}C\\ \mathbf{0}_3\end{pmatrix},\ t_4=0$$

e.g. for URM5, $n=5$

$$(13.5a)\ \mathbf{M}=\begin{pmatrix}C & 0 & 0\\ 0 & C & 0\\ \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_{33}\end{pmatrix},\ j=1,\ i=2,\ k=1\ldots 5,\ \mathbf{X}'_{501,k}=\mathbf{M}_{k2}$$

(13.5b) $\mathbf{X}'_{501,k} = \begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix}_k$, $\mathbf{M}_{k2} = \mathbf{X}'_{501} \sim \mathbf{X}_{50B}(t_4 = 0)$

$j = 2$, $i = 1$, $k = 1 \ldots 5$, $\mathbf{X}'_{502,k} = \mathbf{M}_{k1}$

(13.5c) $\mathbf{X}'_{502,k} = \begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix}_k$, $\mathbf{M}_{k1} = \mathbf{X}'_{502} \sim \mathbf{X}_{50C}(t_5 = 0)$.

Some points on this general solution are made following:

(12.10) The two vectors $\mathbf{X}_{n+}$ (13.2a) and $\mathbf{X}_{n0A}$ (13.2c) remain, effectively, the URM3 vectors $\mathbf{X}_{3+}$ (B2a) and $\mathbf{X}_{30}$ (B2b), embedded as the last three elements within their n-dimensional counterparts, and padded with $n-3$ leading zeros.

(13.6) All expressions for arbitrary dimension n (including $n = 3$) are quadratic only in the evolutionary time; there are no higher order terms. Furthermore, it is only the $\mathbf{X}_{n-}$ vector (13.2b), physically equated with a position (2.1), which contains this quadratic term.

(13.7) Each additional dimension adds a new zero eigenvalue and associated eigenvector, e.g. $\mathbf{X}_{40B}$ (8.1d) for URM4, and $\mathbf{X}_{50C}$ (11.1e) for URM5.

(13.8) Eigenvector $\mathbf{X}_{n-}$ generally always comprises n non-zero elements. Because such vectors are Pythagorean n-tuples, the Pythagoras equation acts as a constraint, and so the n elements occupy an $n-1$, discrete hypersurface of the n-dimensional space. Hence the geometric interpretation of $\mathbf{X}_{3-}$ and $\mathbf{X}_{3+}$ as 2D, discrete cones in URM3's 3D space (lattice), see [1],#3 and [3]. Note too that $\mathbf{X}_{n0A}$ (13.2c) is equivalent to $\mathbf{X}_{30}$, and $\mathbf{X}_{30}$ forms the discrete hyperboloid in URM3.

(13.9) The zero vectors $\mathbf{X}_{n0B}$, $\mathbf{X}_{n0C}$ etc., always comprise four or less (usually always exactly four), non-zero elements, but never more than four in any arbitrary dimension n, i.e. they contain at least $n-4$ zero elements and generally occupy a 4D subspace of URMn. The missing zero vector, $\mathbf{X}_{n0A}$ (13.2c), in this point, is mentioned above in point (13.8).

(13.10) The zero vectors $\mathbf{X}_{n0A}$, $\mathbf{X}_{n0B}$, $\mathbf{X}_{n0C}$ etc., with their reciprocals $\mathbf{X}^{n0A}$, $\mathbf{X}^{n0B}$ and $\mathbf{X}^{n0C}$ etc., Appendix (E), satisfy the same, hyperbolic DCE (5.7) as per URM3, which is given by the scalar product $\mathbf{X}^{n0} \cdot \mathbf{X}_{n0}$, see Appendix (F10).

(13.11) Each zero vector $\mathbf{X}_{n0A}$, $\mathbf{X}_{n0B}$, $\mathbf{X}_{n0C}$ etc., is implicitly parameterised in terms of the URM3, three-parameter solution k, l (A3d) and t_3 (A4c), by virtue that they all embed the URM3 vector $\mathbf{X}_{30}$ (A8b). However, eigenvalue C is also present in all these vectors, except

$\mathbf{X}_{n0A}$ (13.2c), and acts as an initialisation parameter, dictating the total conserved energy (per unit mass) of the space, as in $E = C^2$ (2.1).

Continuing with the analysis of the nth dimension solution, for large evolutionary times t_n, identical remarks made for the URM4 and URM5 solutions also apply to the general case URMn. Therefore, generalising these remarks to URMn:

The only time-dependent (t_i, $i = 4 \ldots n$) vectors with a non-zero contribution to the excess dimensions are $\mathbf{X}_{n-}$ and the zero eigenvectors $\mathbf{X}_{n0j}$, $j = 1 \ldots n-3$, $\mathbf{X}_{n01} \sim \mathbf{X}_{n0B}$, $\mathbf{X}_{n02} \sim \mathbf{X}_{n0C}$ etc, i.e. all but $\mathbf{X}_{n+}$ and $\mathbf{X}_{n0A}$ ($\sim \mathbf{X}_{n0j}, j = 0$). The zero eigenvectors $\mathbf{X}_{n0j}$ are also naturally separated into one vector for each dimension.

There are no mixed terms comprising products of two or more evolutionary times, e.g. $t_4 t_5$, and $\mathbf{X}_{n-}$ can be decomposed into independent terms in each evolutionary time t_i, hence the summation form of $\mathbf{X}_{n-}$ (13.2b).

Not only does the URMn solution separate into a vector term for each unique, excess dimension, but also the terms are identical, when disregarding the particular, excess dimension. They are identical in so far as the rth dimensional component ($r = 4 \ldots n$) contributes a linear term $2t_r C$ in $\mathbf{X}_{n-}(t_r)$, and C in $\mathbf{X}_{n0(r-3)}(t_r)$, and the s dimensional component ($s = 4 \ldots n$, $r \neq s$) contributes a term $2t_s C$ in $\mathbf{X}_{n-}(t_s)$ and C in $\mathbf{X}_{n0(s-3)}(t_s)$, which are identical upon interchange of times t_r and t_s. Since it is only the magnitude of the contribution in the excess dimension that matters in the analysis of the compactification ratio χ, Section (6), each dimension can be treated separately and, furthermore, identical findings for one specific dimension apply to all the others. The separability of the URMn solution into unique dimensional components, and the interchange symmetry between any two components (as regards ratio χ), means that each dimension acts independently of the other, with identical behaviour for equal evolutionary times; the same general expression for χ being used in each case.

The solution for a single, excess rth dimension can be isolated from all the other $n-4$ dimensions s, where $s = 4 \ldots n$, $s \neq r$, by setting all other evolutionary times to zero, i.e. $t_r > 0$, $t_s = 0$.

With these points in mind, then generalising to a specific rth dimension, $r = 4 \ldots n$, evolutionary time t_r, for sufficiently large times t_r (10.15), the rth dimension's compactification ratio χ_r is given in an exact and approximated form as follows, which are simply relabelled versions of (6.3) and (6.7) respectively,

(13.12) $\chi_r = \dfrac{|\mathbf{X}|_r}{|\mathbf{X}|_3}$, exact form, (6.3)

$$\chi_r \approx \frac{2C}{t_r |\mathbf{X}_{3+}|}, \text{ (6.7)}, \ t_r \neq 0, \text{ use (6.3) when } t_r = 0.$$

Since χ_r is inversely proportional to time t_r, it will converge to zero for all times $t \geq t_r$ and, by preceding arguments in Section (9) onwards, the entire solution compactifies to URM3.

To summarise, when any one or more evolutionary times in any excess dimension, is sufficiently large, the n-dimensional vector space compactifies to that of the eigenvector space of URM3. Specifically, all n-dimensional eigenvectors align with the single, static URM3 eigenvector $\mathbf{X}_{3+}$, which occupies a 2D conical subspace of URM3, hence URMn compactifies to appear two-dimensional. The conical subspace is actually a 2-parameter, discrete surface termed the 'cone' in URMT (footnote 10).

This completes the general analysis of compactification for an n-dimensional space in URMT.

14 Answers to Anticipated Questions

Before summarising and concluding the paper, a few anticipated questions are answered.

(14.1) **Why not set all evolutionary times equal?**

As currently formulated, n-dimensional URMT has $n-2$ temporal parameters controlling the evolution, of which $n-3$ parameters control the evolution of each of the $n-3$ excess dimensions, one for each dimension; the last three dimensions, that of URM3, share the single evolution parameter t_3. As concluded, if any one or more of these $n-2$ parameters is sufficiently large (Section (10)), then the entire n-dimensional eigenvector space compactifies to a two dimensional, conical subspace of URM3, that of eigenvector $\mathbf{X}_{3+}$. Because of this, there is no compelling reason to make them all the same parameter, as doing so would only hasten the compactification, but not change the final result - with one small exception: if only the jth parameter of an excess dimension were made large, and all others, say, relatively small or zero, i.e. $t_j >> t_i, i = 4 \ldots n-3, i \neq j$, then the jth dimension would appear relatively larger than all other excess dimensions, by a factor of t_j / t_i, because the jth dimension is of size $t_j C$ and the ith dimension of size $t_i C$. Of course, relative to URM3 and $\mathbf{X}_{3+}$, and under the sufficiency condition (14.82), the jth dimension would still appear small. Thus, this exception might be justification to set all evolutionary parameters equal so that all excess dimensions appear of the same relative size; this issue remains open.

(14.2) **Does it have to be in integers?**

Integers are used throughout URMT but, as regards compactification, it is not currently known whether this is strictly necessary. Certainly URMT in [1],#1 can go quite a long way before integers are required, and then they only enter when gcd conditions are imposed, which is after transformation invariance is imposed. Provided some form of quantisation is mandated, it may well be feasible to broaden the compactification aspects to the real and complex domains. Nevertheless, URMT is currently formulated entirely in integers, and therefore the mathematics of URMT compactification is also formulated in integers. It is anticipated that complex integers may well enter in further development of the theory, but this matter is still in its infancy.

(14.3) **Why stop at URM3, why not URM2 or lower?**

Firstly, there is no meaningful URM1, but URM2 is perfectly plausible, see Appendix (H).

The compactification has been shown to terminate at the 2D conical subspace, represented by $\mathbf{X}_{3+}$, of URM3's three-dimensional eigenvector space. However, can the compactification continue within URM3 down to URM2?

The answer is yes, if a trivial (I7) initial solution for $\mathbf{X}_{3+}$ is acceptable, and no, if unacceptable - which is the eventual answer, i.e. it isn't acceptable on the grounds of being too simplistic. However, assuming yes for a while, such a trivial solution would be a vector $\mathbf{X}_{3+} = (0,1,1)^T$, where the x coordinate in $\mathbf{X}_{3+}$ is zero and $y = 1$, $z = 1$, giving the trivial

Pythagorean twin $(1,1)$ where $1^2 = 1^2$! This twin pair $(1,1)$ is the realm of URM2, and the solution $\mathbf{X}_{3+} = (0,1,1)^T$ is basically a URM2 'lifted' (I6) solution $\mathbf{X}_{2+} = (1,1)^T$ (the only such non-primitive (I8) solution) embedded within URM3. Whilst this pair $(1,1)$ is considered rather simple, it can actually be used as the starting point to generate 3D and higher solutions. In other words, URMT can start at URM2 and work upward, just like URM3 works upward to URM4 etc. However, because $\mathbf{X}_{2+} = (1,1)^T$ is the only primitive, non-trivial solution, it is not even 2D or even 1D, but just a point in space. A non-primitive solution such as $\mathbf{X}_{2+} = (y, y)^T$, parameter y, is mathematically acceptable, giving a 1D, straight line solution, but then the gcd criterion (3.4c) on $\mathbf{X}_{2+}$ is not satisfied since y is a common factor of both elements. More importantly, even this solution is limited, and 'lifting' it to 3D (I6) only gives a subset of the full URM3 formulation.

As a consequence of the above, URM2 is considered too limited to be of physical use, at least at present, i.e. it has not been rejected, and its place in URMT is left undecided - it does have a beauty in its simplicity, but perhaps just a bit too simple. A good reason not to dismiss it lightly is that does not (cannot) have its own evolutionary parameter t_2, and cannot therefore compactify from two to one dimension, at least not by the growth of an evolutionary parameter. Secondly, its only free parameter is the eigenvalue C, which, as noted (2.1), relates to the total conserved energy $E = C^2$. Eigenvalue C is an initial condition since it is conserved, i.e. invariant to arbitrary variations in all n free parameters of URMn (when under Pythagoras conditions - there are many more when not under these conditions).

Knowing there is the capability to reduce to URM2, the next question is, why doesn't this appear in the URM3 evolution equations, Appendix (B)?

Basically, the triad of URM3 eigenvectors flattens (or aligns), in the large evolutionary limit, $t_3 \to \infty$, to align with $\mathbf{X}_{3+}$, see (I5), but none of the elements of the vector $\mathbf{X}_{3+}$ shrink relative to each other because t_3 scales every element of the vector $\mathbf{X}_{3+}$ equally, and every element x, y, z of $\mathbf{X}_{3+}$ is trivially non-zero. The vector $\mathbf{X}_{3+}$, as for all higher dimensional forms $\mathbf{X}_{n+}$, is static, i.e. not a function of any evolutionary time t_j, but it is still a function of two free parameters, k and l. Whilst a subset of solutions for $\mathbf{X}_{3+}$ could have one of k or l zero, but not both (A3d), generally neither are zero and $\mathbf{X}_{3+}$, therefore, comprises three non-zero elements, i.e. it is classed as non-trivial. Alternatively stated, none of the two elements x or y of $\mathbf{X}_{3+}$ is always zero, and z is never zero by Pythagoras (F1).

The non-triviality of $\mathbf{X}_{3+}$ is the absolute key as to why the example 5D formulation, vector $\mathbf{X}_{5+}$ (5.2) embeds $\mathbf{X}_{3+}$ as x, y, z, but is zero in its fourth and fifth dimensional elements (the first two elements of the vector). In effect, it is an embedding of a non-trivial 3D formulation, i.e. that of URM3. Conversely, a non-trivial 4D embedding, i.e. four non-zero elements w, x, y, z, would compactify to URM4 but not compactify further to URM3.

Concluding the above, the compactification can be made to stop with URM2 according to the initial conditions, i.e. by arranging the number of non-zero elements in the lowest dimension of $\mathbf{X}_{n+}$, but with the simplest and only primitive, 2D solution, $\mathbf{X}_{2+} = (1,1)^T$, it would compactify to the point $(1,1)$ - perhaps this really is telling us something! In the work presented herein, it stops at three by virtue of three non-zero elements x, y, z in $\mathbf{X}_{3+}$. Therefore, the biggest reason not to stop at URM2 is one of too much simplicity and not enough complexity. Mathematically this is fine, but physically it is rather simplistic. However, this raises the next question.

(14.4) **Why not stop at URM4 or higher?**

From the arguments at the end of the previous question, it would seem that setting all four elements w, x, y, z of $\mathbf{X}$ (3.4a) non-zero, and the first element v to zero, is sufficient to stop the compactification process at URM4 instead of URM3.

First and foremost, there is a strong case against this given the following physical constraints placed upon the solutions.

(14.4.1) The solution for $\mathbf{X}_{n+}$ must be invariant to all arbitrary variations in all evolutionary parameters - this is a URMT imposed, general constraint arising from the 'Invariance Principle' in [1],#1, which is essentially a postulate of URMT, see [3].

(14.4.2) The Potential V must be zero and invariant for all arbitrary variations in all parameters. In URM4 and URM5, at least, this condition makes for only two, non-zero eigenvalues $\pm C$, with all others zero; see [4]. In essence, an invariant, zero Potential solution is desirable from a physical viewpoint as it represents a constant energy, force-free trajectory through the eigenvector 'lattice', see [1],#3 and 3. To satisfy this invariant, zero Potential constraint, there are the two further constraints, (14.4.2a) and (14.4.2b) below.

(14.4.2a) The matrix $\mathbf{A}_{n0}$ must satisfy Pythagoras conditions - primarily for Minkowski, metrical reasons and, physically, URMT must accommodate STR somewhere; see also footnote 24.

(14.4.2b) The matrix $\mathbf{A}_{n0}$ must have a certain form whereby all the rows and columns, barring the last three (URM3) or four (URM4), are multiples of the eigenvectors $\mathbf{X}^{(n-1)-}$ (rows) and $\mathbf{X}_{(n-1)+}$ (columns) see Appendix (C). This latter point is subtle and discussed more fully in [4]. It is very important though for the purposes of obtaining a quadratic expression for the eigenvectors. This then leads to the highest order terms in any expression being a quadratic function of the evolutionary parameters $t_j, j = 3 \ldots n$, and making all scalar invariants quadratic in the eigenvalue, i.e. C^2, which is very important to physically associate the general DCE (F10) with an energy conservation equation, irrespective of dimension n.

To satisfy the above conditions for URM3 is relatively easy and leads to an infinite set of solutions for eigenvectors $\mathbf{X}_{3-}$ and $\mathbf{X}_{30}$, parameterised by t_3 (m or δ in [1]-[3]), for every static eigenvector $\mathbf{X}_{3+}$. As noted, $\mathbf{X}_{3+}$, itself, is a full, two-parameter family, parameters k and l, see footnote 7.

All these above points are very important because it seems to be all change in URM4. Taken together, the above conditions, (14.4.1) and (14.4.2), are severe and, to meet them in URM4, it seems there is only one solution for $\mathbf{X}_{4+}$. It is actually considered remarkable that there is exactly one solution[44] and not two or more, or even no solutions - just one single solution? Even more curious, this solution is the simplest it could possibly be, namely the Pythagorean quadruple $(2,1,2,-3)$, with strict adherence to the sign of its elements. It also only arises by adding in a lot of simplifications. Whilst this particular solution can be shown to be the only solution, under its 'PS+RU' parameterisation scheme, no general proof that URM4 cannot satisfy the above conditions, without more simplifications, exists. The most notable simplification is to change $\mathbf{X}_{4+}$ from four non-zero elements to three, and thereby reduce the theory to that of URM3, under the above conditions. Whilst no proof is offered, the author has not found any general solution, of any worth, without having to add too many simplifications, or reducing URM4 back to URM3.

With four-dimensional STR in mind, it might seem nicer to stop at URM4. This would mean all four eigenvectors of URM4 would, generally, be non-trivial (four non-zero elements). However, non-trivial 4D vectors, specifically $\mathbf{X}_{4-}$, can already be generated from the URM3 formulation, as can higher, n-dimensional vectors $\mathbf{X}_{n-}$, by embedding URM3 within URMn. Given $\mathbf{X}_{n-}$ can be physically associated with a position vector, it would seem URM3 will suffice, at least spatially speaking, to give a four-vector position. Of course, that's just position, and not the only four-vector, so it is nicer to have a bit more flexibility, which is why URMT was originally extended from its 3D origins in [1].

From another physical perspective though, nature is generally ternary, everything (almost) appears to come in threes, from spatial dimensions to families of particles. It is also of note that URM5 (a favourite of the author and a reason behind its usage in this paper), has three evolutionary time parameters, t_3, t_4 and t_5, which makes a nice symmetric triplet to go with the spatial dimensions of URM3. Admittedly this is aesthetics; after all, this does not include the laboratory time t, which is considered to be an interval (in URMT anyhow), not an absolute evolutionary time. The laboratory time is conjectured to hide in the last element of $\mathbf{X}_{4-}$; see footnote 24.

[44] This solution is cryptically known as the 'PS+RU' solution in [4] because of its condition PS+RU=0 on the dynamical variables, P, R, S and U.

However, the original URMT formulation in [1] seems at its best for URM3, at least in terms of results, physical interpretation and tractability. It has three sets of standard variables, plus their conjugate and dual forms, see [1],#5. Additionally, the dynamical variables P, Q, R, and their conjugates $\overline{P}, \overline{Q}, \overline{R}$, are isomorphic with the complex roots of unity, whereas in the most general, n-dimensional formulation of URMT, the dynamical variables, e.g. those (3.2) in the URM5 $\mathbf{A}_{50}$ matrix (3.1), lose this complex nature. This may seem a whim, but it is a nod to the fact that mathematical physics uses complex numbers as a given, and it is felt URMT requires an equivalent algebra in its discrete formulation. So, whatever n-dimensional URMT formulation is used, it is desirable it to encompass URM3 as a subset, and this is one reason why the matrix $\mathbf{A}_{30}$ (A1a) is embedded in $\mathbf{A}_{50}$ (3.1).

To conclude these questions, starting the general, n-dimensional URMT formulation at URM3 seems to offer the best combination of being able to extend to any number of dimensions, whilst retaining all physical properties of URM3. Although stopping the formulation at URM3 means that compactification also stops at two spatial dimensions, it also ensures that it doesn't descend into triviality, i.e. too much physical simplicity.

(14.5) One last question

If a velocity eigenvector, e.g. $\mathbf{X}_{50B}$, grows linearly with time, by virtue of a constant acceleration $\mathbf{X}_{3+}$, as in $\mathbf{X}_{50B} = -t_4 \mathbf{X}_{3+}$ (11.1d), will it not at some stage exceed the speed of light, little c?

Strictly yes, but this is a 'space' and it is not physically clear what the expansion limit of the space is or, indeed, whether it has a physical presence (for want of a better term). It could simply be that the acceleration is so tiny that the evolutionary time has to be enormous to compensate, and no such evolutionary stage has yet been reached. Going back to URMT's roots in URM3 [1], the space is basically a discrete, infinite set of points, i.e. the eigenvector space, also known as a lattice, and it is the underlying space upon which it is thought that physics plays-out as functions on this lattice.

15 Summary

In brief, the entire compactification process shows that the first three dimensions expand relative to the excess dimensions, making the excess dimensions appear to shrink over time with respect to the first three. However, the expansion in the first three dimensions is in a particular vector direction, characterised by two, free parameters. Hence the compactification is said to converge to a two-dimensional subspace of the first three dimensions.

Note, all quantities are in integers; all spaces are discrete sets of points.

Before commencing, the URM5, five-dimensional eigenvector solution is reproduced below, and recommended for a quick visualisation of the points made.

The URM5 eigenvector solution, reproduced from Section (11),

(11.1)

(11.1a) $$\mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} \text{static, acceleration vector}$$

(11.1b) $$\mathbf{X}_{5-} = -(t_5^2 + t_4^2)\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_5\begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + 2t_4\begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3-} \end{pmatrix}$$

evolving, position vector

(11.1c) $$\mathbf{X}_{50A} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{30} \end{pmatrix} \sim \mathbf{X}_{50}\text{, evolving velocity vector } \mathbf{X}_{30}$$

(11.1d) $$\mathbf{X}_{50B} = -t_4\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix} \sim \mathbf{X}_{501}\text{, evolving velocity vector}$$

(11.1e) $$\mathbf{X}_{50C} = -t_5\begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} \sim \mathbf{X}_{502}\text{, evolving velocity vector}$$

It is important to note that the entire paper focuses on the evolution of the eigenvectors relative to those of URM3. Keep in mind that the URM3 eigenvectors also evolve in an identical fashion, all converging on the single $\mathbf{X}_{3+}$ vector, see Appendix (B). It is chiefly this reason that it is not necessary to delve into URM3's eigenvector evolution herein; the compactification conclusions are the same. URM3's geometric evolution, in terms of 'flattening', cones and hyperboloids, is fully covered in [1],#3 and summarised in [3].

Eigenvectors and Eigenvalues

The general, n-dimensional unity root matrix theory, URMn, $n \geq 3$, for a square $n \times n$ matrix $\mathbf{A}_{n0}$, has n linearly independent eigenvectors, each with n elements. The first element is the nth dimensional coordinate, and the last three elements are the first three dimensions, and functions of the three URM3 eigenvectors $\mathbf{X}_{3+}$ (acceleration), $\mathbf{X}_{30}$ (velocity) and $\mathbf{X}_{3-}$ (position).

The eigenvectors split into three types, as per URM3, a single acceleration vector $\mathbf{X}_{n+}$ (eigenvalue C), a single position vector $\mathbf{X}_{n-}$ (eigenvalue $-C$), and $n-2$ velocity vectors $\mathbf{X}_{n0j}$, $j = 0 \ldots n-3$ (for $n-2$ zero eigenvalues).

The entire vector space is characterised by $n-2$ temporal parameters t_j, $j = 3 \ldots n$, and two non-temporal, k and l, i.e. n parameters in total.

The first three dimensions (last three elements) have a single evolutionary time parameter t_3, the remaining $n-3$ excess dimensions each have their own temporal parameter.

Each temporal parameter emerges from an invariance transformation on the elements of the unity root matrix $\mathbf{A}_{n0}$, which leaves the eigenvector equation $\mathbf{A}_{n0}\mathbf{X}_{n+} = C\mathbf{X}_{n+}$ invariant to any arbitrary variation in these parameters.

The other two, non-temporal parameters k and l, control the URM3 vector solution, most notably, the acceleration $\mathbf{X}_{3+}$ and, hence, also $\mathbf{X}_{n+}$.

Other than these n parameters (k l, t_j, $j = 3 \ldots n$,), there is a single, fundamental constant, the eigenvalue C (big C), which is the only contributor to all excess dimensions, excepting their respective evolutionary, temporal parameters, which also affect the first three dimensions.

The constant eigenvalue C is equated with a scalar velocity (speed), and controls the size of all excess dimensions. It is related to the total conserved energy $E = C^2$, and is the single tuning constant, dictating the initial, time-zero, expansion velocity $\mathbf{X}_{n0j}$ of the excess dimensions, $j = 1 \ldots n-3$.

Eigenvector evolution

The first three dimensions (last three elements) of the acceleration vector $\mathbf{X}_{n+}$ (13.2a) are just the URM3 acceleration vector $\mathbf{X}_{3+}$, which is a non-trivial (three non-zero elements), static (no evolutionary time-dependence) acceleration vector.

The first three dimensions of the jth velocity vector $\mathbf{X}_{n0j}$, $j = 1 \ldots n-3$ (13.2d), are a linear function in time t_{j+3} of the URM3 acceleration vector $\mathbf{X}_{3+}$, i.e. the integral of acceleration to give velocity, $\mathbf{X}_{n0j} = -t_{j+3}\mathbf{X}_{n+}$. This is also true for the URM3 velocity vector $\mathbf{X}_{30}$ scaled by t_3, i.e. $j = 0$.

The first three dimensions of the position vector $\mathbf{X}_{n-}$ (13.2b) are a quadratic function in time of the URM3 acceleration vector $\mathbf{X}_{3+}$, with a contribution for each excess temporal coordinate t_j, $j = 1 \ldots n-3$, i.e. the sum $\mathbf{X}_{n-} = \sum -t_{j+3}^2 \mathbf{X}_{3+}$. The magnitude of this growth is approximated by $\left|t_{j+3}^2 \mathbf{X}_{3+}\right|$, e.g. $\left|t_5^2 \mathbf{X}_{3+}\right|$ (11.1b), for one specific evolutionary time t_j, a summation more appropriate for evolution in two or more dimensions - see (9.10) pr (10.3a). This is also true for the URM3 velocity vector $\mathbf{X}_{3-}$ scaled by t_3^2, i.e. $j = 0$.

All excess dimensions (the first $n-3$ elements) of the acceleration vector $\mathbf{X}_{n+}$ (13.2a), are all zero, i.e. there is no acceleration (dimensional growth) in the excess dimensions.

The jth excess dimension (element $n-(j+2)$) of each velocity vector $\mathbf{X}_{n0j}$, $j = 1 \ldots n-3$ (13.2d), comprises the constant eigenvalue C only, with no time-dependence, i.e. static in all excess dimensions. Therefore, the excess dimensions of the velocity vector remain a constant 'size', i.e. constant velocity (or speed) C.

The jth excess dimension of the position vector $\mathbf{X}_{n-}$ (13.2b) is a linear function in evolutionary time t_j of the velocity constant C, i.e. the integral $2t_jC$, e.g. $\left|2t_5C\right|$ in (11.1b).

All the above eigenvector integrals are really just Newton II, to within a sign and scale factor. But note that the excess dimensions are a first integral of the constant velocity C, whereas the first three dimensions are first and second integrals of a constant acceleration $\mathbf{X}_{3+}$.

Of all these eigenvector growth rates, only the first three dimensions of the position vector $\mathbf{X}_{n-}$ have a quadratic dependence on time. There is no quadratic time contribution in any excess dimensions, only linear. All quadratic growth is therefore in the $\mathbf{X}_{n-}$ vector, which aligns, over time, in the direction of the $\mathbf{X}_{3+}$ acceleration vector, embedded in the first three dimensions, i.e. the home of URM3.

Although URM3's own explicit evolution has not been detailed, see further above, the URM3 eigenvector evolution equations in Appendix (B) show identical evolutionary behaviour for all URM3 eigenvectors; in particular, quadratic growth with respect to time t_3 in the $\mathbf{X}_{3-}$ vector and in the direction of the acceleration $\mathbf{X}_{3+}$.

Summarising the above: for any single, sufficiently large (Section (10)), evolutionary time t_j, where t_j is any one of the $n-2$ evolutionary time parameters, the entire n-dimensional vector space converges on the $\mathbf{X}_{n-}$ position vector, which, itself, aligns and grows along URM3's $\mathbf{X}_{3+}$ acceleration vector. Simultaneously, the excess dimensions continue to grow, albeit as a linear function of time with a constant velocity C in each excess dimension.

Comparing the linear growth of the jth excess dimension $2|t_j C|$, for a specific evolutionary time t_j, with the simultaneous, quadratic growth $|t_{j+3}^2 \mathbf{X}_{3+}|$ of the first three dimensions, gives a measure of the relative size of the jth excess dimension with respect to the first three dimensions. This measure is termed the compactification ratio of the jth dimension, denoted by χ_j and approximated as follows; see Section (6), (6.3) for the exact form for χ_j.

(6.7) $\chi_j \approx \dfrac{2C}{t_j|\mathbf{X}_{3+}|}$, the compactification ratio of the jth dimension.

Note that χ_j is dimensionless, and the ratio is inversely proportional to the evolutionary time t_j such that it limits to zero, i.e.

(6.4) $\lim_{t_j \to \infty} \chi_j = 0$.

The relative error ε (10.2) in the approximation also limits to zero, i.e. the approximation gets better with increasing time t_j.

As a consequence of χ_j limiting to zero then, over a sufficiently large evolutionary time t_j, the size of the excess, jth dimension appears to shrink into insignificance with respect to the first three dimensions; concurrent growth in any other excess dimension only hastening the compactification.

Lastly though, the 'first three dimensions' are really just the single direction of the acceleration vector $\mathbf{X}_{3+}$. Whilst this might seem to be a compactification to one dimension, the vector $\mathbf{X}_{3+}$ is arbitrarily specified by two other, non-temporal parameters k and l. In fact, $\mathbf{X}_{3+}$ is actually a Pythagorean triple, where the two parameters form the standard Pythagorean parameterisation. Thus, $\mathbf{X}_{3+}$ is really a 2D, discrete, conical surface, described as two cones, 'upper' and 'lower', in URMT [1],#3, [2], [3].

16 Conclusion

The n-dimensional, discrete eigenvector space of Unity Root Matrix Theory appears to reduce its dimensionality, i.e. compactify, as its temporal evolution progresses, to a two-dimensional, discrete, conical surface embedded within a three-dimensional, discrete, eigenvector space. The conical surface is formed from the elements of a two-parameter, static acceleration eigenvector, to which all eigenvectors align in the limit as the evolutionary time, in one or more dimensions, tends to infinity.

17 Acknowledgements

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18 References

[1] Unity Root Matrix Theory, Physics in Integers, R J Miller, FastPrint Publishing, 2011, ISBN 978-184426-974-7, http://www.fast-print.net/bookshop/823/unity-root-matrix-theory-physics-in-integers

This book is broken into six separate papers, each paper is given a specific reference #1 to #6 as follows:

[1],#1 Unity Root Matrix Theory Foundations
[1],#2 see [2], below
[1],#3 Geometric and Physical Aspects
[1],#4 Solving Unity Root Matrix Theory
[1],#5 Unifying Concepts
[1],#6 A Non-unity Eigenvalue

[2] Pythagorean Triples as Eigenvectors and Related Invariants, www.urmt.org. R J Miller, 2010. A free PDF available for download.

[3] Unity Root Matrix Theory, Overview. Presentation_URMT_Shortform.doc, www.urmt.org. R J Miller, Issue 1.1, 09/07/2011. This presentation is an overview of the six papers, published in [1]. A free PDF available for download.

[4] Unity Root Matrix Theory, Higher Dimensional Extensions, R J Miller, FastPrint Publishing, 2012, ISBN 978-178035-296-1, http://www.fast-print.net/bookshop/1007/unity-root-matrix-theory-higher-dimensional-extensions

[5] Sadri Hassani, Foundations of Mathematical Physics, Prentice-Hall International Editions, 1991, ISBN 0-13-327503-5.

[6] I Niven, S Zuckerman, H L Montgomery, An Introduction to the Theory of Numbers, 5th Edition, John Wiley & Sons, Inc 1991. ISBN 0-471-54600-3.

19 Appendix (A) URM3 Background Information

This Appendix provides some basic background information on URM3. For a full account, see [1], [2] and [3].

The general, unity root matrix $\mathbf{A}_3$, comprising 'dynamical variables' P, Q, R and their conjugates $\overline{P}, \overline{Q}, \overline{R}$, is defined as

(A1)

(A1a) $\mathbf{A}_3 = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix}$

(A1b) $P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0)$

(A1c) $\overline{P}, \overline{Q}, \overline{R} \in \mathbb{Z},\ (\overline{P}, \overline{Q}, \overline{R}) \neq (0,0,0)$

(A1d) Notation

$\mathbf{A}_3 \sim \mathbf{A}$ in [1] for general URM3

$\mathbf{A}_3 \sim \mathbf{A}_{30} \sim \mathbf{A}_0$ in [1],#2 under URM3 Pythagoras conditions, below.

An eigenvector $\mathbf{X}_{3+}$ to matrix $\mathbf{A}_{30}$, for eigenvalue C, is defined as

(A1e) $\mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$x, y, z \in \mathbb{Z},\ (x, y, z) \neq (0,0,0)$

(A1f) $\mathbf{A}_{30}\mathbf{X}_{3+} = C\mathbf{X}_{3+},\ C \in \mathbb{Z},\ C \geq 1$

(A1g) Notation

$\mathbf{X}_{3+} \sim \mathbf{X}$ in general URM3

$\mathbf{X}_{3+} \sim \mathbf{X}_+$ when under URM3 Pythagoras conditions, below.

The Pythagoras conditions on the dynamical variables are

(A2a) $\overline{P} = P$, $\overline{Q} = Q$, $\overline{R} = -R$,

and the matrix $\mathbf{A}_3$ becomes

(A2b) $\mathbf{A}_{30} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix}$, see (A1d) notation

There are now three distinct eigenvalues

(A2c) $\lambda = +C$, $\lambda = 0$, $\lambda = -C$,

and, consequently, two additional eigenvectors $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$, defined by their eigenvector equations as

(A2e) $\mathbf{A}_{30}\mathbf{X}_{30} = 0$, $\lambda = 0$
(A2f) $\mathbf{A}_{30}\mathbf{X}_{3-} = -C\mathbf{X}_{3-}$, $\lambda = -C$.

URM3, under Pythagoras conditions (A2a), is a completely solved problem with an analytic solution for all variables. The x, y, z coordinates forming $\mathbf{X}_{3+}$ (A1e) are parameterised by two, arbitrary integers k and l.

(A3)
(A3a) $x = 2kl$
(A3b) $y = (l^2 - k^2)$
(A3c) $z = (l^2 + k^2)$
(A3d) $k, l \in \mathbb{Z}$, $(k, l) \neq (0,0)$, $\gcd(k, l) = 1$

The scale factors $\alpha_3, \beta_3, \gamma_3$ (A6), (I2), and dynamical variables P, Q, R (A5) are obtained by solving the following linear Diophantine equation using Euclid's algorithm [6], for unknown integers s and t[45], given k and l

(A4) $+C = ks - lt$, $s, t \in \mathbb{Z}$.

Solving this very simple equation introduces some indeterminacy into URM3 in an otherwise, completely deterministic, analytic solution. It has to be solved algorithmically, with no analytic solution. Physically, this indeterminacy is very likely a good thing; without it, the

[45] The usage of symbol t here is slightly unfortunate in that it is not a time parameter. It is retained for compatibility with all existing URMT literature. In fact, it is shown in [1] that t has units of $\sqrt{L}$, but since it always multiplies k or l, with units of $\sqrt{LT^{-1}}$, or appears in squared form, then potentially irrational quantities do not appear in the solution, in keeping with one of the URMT postulates, 'all observables are integers', see [3].

entire n-dimensional URMT solution would be completely deterministic once initial conditions are imposed.

To obtain a particular solution s' and t', this equation has to be solved algorithmically. Once a particular solution is obtained, then an infinite family of solutions is obtained, denoted by integers s and t, and parameterised by another arbitrary integer t_3. This parameter t_3, is none other than the URM3 evolutionary time.

(A4a) $s = s' + t_3 l$

(A4b) $t = t' + t_3 k$

(A4c) $t_3 \in \mathbb{Z}$, notation $t_3 \sim m, \delta$ in [1]

(A4d) $s', t' \in \mathbb{Z},\ (s', t', t_3) \neq (0,0,0)$

The dynamical variables P, Q, R are parameterised in terms of k, l and t_3, implicitly via the general solutions for s (A4a) and t (A4b), as follows, and likewise for the URM3 scale factors[46] (I10)

(A5)

(A5a) $P = -(ks + lt)$

(A5b) $Q = (ls - kt)$

(A5c) $R = -(ls + kt)$

(A6)

(A6a) $\alpha_3 = -2st$, $\alpha_3 \sim \alpha$ in [1]

(A6b) $\beta_3 = (t^2 - s^2)$, $\beta_3 \sim \beta$ in [1]

(A6c) $\gamma_3 = (t^2 + s^2)$, $\gamma_3 \sim \gamma$ in [1]

(A7)

The following table gives the solutions in, all variables, for eigenvalue $C = 1$ and $t_3 = 0$, for a few small values of the parameters k and l.

l	k	x	y	z	s	t	P	-Q	R	α_3	β_3	$-\gamma_3$
2	1	4	3	5	1	0	-1	-2	-2	0	-1	-1
3	2	12	5	13	2	1	-7	-4	-8	-4	-3	-5
4	1	8	15	17	1	0	-1	-4	-4	0	-1	-1
4	3	24	7	25	3	2	-17	-6	-18	-12	-5	-13
5	2	20	21	29	3	1	-11	-13	-17	-6	-8	-10
5	4	40	9	41	4	3	-31	-8	-32	-24	-7	-25
6	1	12	35	37	1	0	-1	-6	-6	0	-1	-1
6	5	60	11	61	5	4	-49	-10	-50	-40	-9	-41
7	2	28	45	53	4	1	-15	-26	-30	-8	-15	-17
7	4	56	33	65	2	1	-15	-10	-18	-4	-3	-5

[46] Note too that the scale factors $\alpha_3, \beta_3, \gamma_3$ form a Pythagorean triple.

7	6	84	13	85	6	5	-71	-12	-72	-60	-11	-61
8	1	16	63	65	1	0	-1	-8	-8	0	-1	-1
8	3	48	55	73	3	1	-17	-21	-27	-6	-8	-10
8	5	80	39	89	5	3	-49	-25	-55	-30	-16	-34
8	7	112	15	113	7	6	-97	-14	-98	-84	-13	-85
9	2	36	77	85	5	1	-19	-43	-47	-10	-24	-26
9	4	72	65	97	7	3	-55	-51	-75	-42	-40	-58
9	8	144	17	145	8	7	-127	-16	-128	-112	-15	-113

The standard eigenvectors $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ are defined in terms of the coordinates x,y,z, dynamical variables P,Q,R, and scale factors $\alpha_3,\beta_3,\gamma_3$ respectively as

(A8)

(A8a) $\mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, (A8b) $\mathbf{X}_{30} = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix}$, (A8c) $\mathbf{X}_{3-} = \begin{pmatrix} \alpha_3 \\ \beta_3 \\ -\gamma_3 \end{pmatrix}$.

The reciprocal eigenvectors $\mathbf{X}^{3+}$, $\mathbf{X}^{30}$ and $\mathbf{X}^{3-}$, (A10) further below, are obtained from the standard forms (A8) using the URM3 $\mathbf{T}_3$ operator relations:

(A9)

(A9a) $\mathbf{T}_3 = \mathbf{T}^3 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

(A9b) $\mathbf{X}^{3+} = \left(\mathbf{T}^3\mathbf{X}_{3-}\right)^T$, (A9c) $\mathbf{X}^{30} = \left(\mathbf{T}^3\mathbf{X}_{30}\right)^T$, (A9e) $\mathbf{X}^{3-} = \left(\mathbf{T}^3\mathbf{X}_{3+}\right)^T$

(A9f) $\mathbf{X}_{3+} = \left(\mathbf{X}^{3-}\mathbf{T}_3\right)^T$, (A9g) $\mathbf{X}_{30} = \left(\mathbf{X}^{30}\mathbf{T}_3\right)^T$, (A9h) $\mathbf{X}_{3-} = \left(\mathbf{X}^{3+}\mathbf{T}_3\right)^T$

(A10)
(A10a) $\mathbf{X}^{3+} = \begin{pmatrix}\alpha_3 & \beta_3 & \gamma_3\end{pmatrix}$
(A10b) $\mathbf{X}^{30} = \begin{pmatrix}P & -Q & -R\end{pmatrix}$
(A10c) $\mathbf{X}^{3-} = \begin{pmatrix}x & y & -z\end{pmatrix}$.

All three sets of variables x,y,z, P,Q,R, $\alpha_3,\beta_3,\gamma_3$ and eigenvalue C are related via the divisibility relations, see [1],#1,

(A11)
(A11a) $C^2 - P^2 = \alpha_3 x$
(A11b) $C^2 - Q^2 = \beta_3 y$
(A11c) $C^2 + R^2 = \gamma_3 z$

It is from these that a clue to the name 'unity roots' arises since they satisfy the following congruences:

(A12)
(A12a) $C^2 \equiv P^2 \pmod{x}$
(A12b) $C^2 \equiv Q^2 \pmod{y}$
(A12c) $C^2 \equiv -R^2 \pmod{z}$.

If the eigenvalue is unity, i.e. $C = 1$, then P, Q, R are the square roots of unity in modulo arithmetic.

(A13)
(A13a) $P^2 \equiv +1 \pmod{x}$
(A13b) $Q^2 \equiv +1 \pmod{y}$
(A13c) $R^2 \equiv -1 \pmod{z}$

20 Appendix (B) URM3 Eigenvector Evolution

The URM3 eigenvector evolution equations are given below, reproduced from [1],#2 where the initial values at URM3 time zero ($t_3 = 0$), are superscripted with a prime. Note that symbol t_3 is denoted by m in [1].

(B1)

(B1a) $\mathbf{X}'_{3+} = \mathbf{X}_{3+}(t_3 = 0)$

(B1b) $\mathbf{X}'_{30} = \mathbf{X}_{30}(t_3 = 0)$

(B1c) $\mathbf{X}'_{3-} = \mathbf{X}_{3-}(t_3 = 0)$.

The evolution equations in standard vector form are:

(B2)

(B2a) $\mathbf{X}_{3+} = \mathbf{X}'_{3+}$, static, no t_3 dependence

(B2b) $\mathbf{X}_{30}(t_3) = -t_3\mathbf{X}_{3+} + \mathbf{X}'_{30}$

(B2c) $\mathbf{X}_{3-}(t_3) = -t_3^2\mathbf{X}_{3+} + 2t_3\mathbf{X}'_{30} + \mathbf{X}'_{3-}$,

and their reciprocal forms

(B2d) $\mathbf{X}^{3-} = \mathbf{X}'^{3-}$, static, no t_3 dependence

(B2e) $\mathbf{X}^{30}(t_3) = -t_3\mathbf{X}^{3-} + \mathbf{X}'^{30}$

(B2f) $\mathbf{X}^{3+}(t_3) = -t_3^2\mathbf{X}^{3-} + 2t_3\mathbf{X}'^{30} + \mathbf{X}'^{+}$.

Notice that $\mathbf{X}_{3-}$ has exactly the same quadratic degree in the evolutionary parameter t_3 as the general, n-dimensional vector $\mathbf{X}_{n-}$ (13.2b).

Given that $\mathbf{X}_{3-}(t_3)$ (B2c), with its quadratic term in t_3, will dominate all three eigenvectors, for sufficiently large t_3 (further below), the eigenvector $\mathbf{X}_{3-}$ will align with $\mathbf{X}_{3+}$ as will $\mathbf{X}_{30}$, i.e.

(B3) $\mathbf{X}_{3-}(t_3) \approx -t_3^2\mathbf{X}_{3+}$,

where the term 'sufficiently large' here, really means any time t_3 such that $t_3^2|\mathbf{X}_{3+}| >> |2t_3\mathbf{X}'_{30} + \mathbf{X}'_{3-}|$ in (B2c). This then depends on the initial conditions for the size of the vectors $\mathbf{X}'_{30}$ and $\mathbf{X}'_{3-}$.

21 Appendix (C) The Residual Matrix Method

The residual matrix method[47] is used to determine eigenvectors in URMT, giving both the standard eigenvector and its reciprocal, e.g. $\mathbf{X}_{n+}$ and $\mathbf{X}^{n+}$ (footnote [48]). It is the same method as used in URM3 [1],#2 and [2] to evaluate the URM3 eigenvectors. The reader is referred to these references for full details since the matrix polynomials, used below, are identical in the determination of the URMn eigenvectors $\mathbf{X}^{n+}$ (from $\mathbf{X}_{n+}$, using $\mathbf{A}_{n0}$), and $\mathbf{X}^{n0}$ (from $\mathbf{X}_{n0}$, using $\mathbf{A}_{n0}$); see note (C19) about obtaining $\mathbf{X}_{n-}$ and $\mathbf{X}^{n-}$.

The residual matrices $\mathbf{E}_{n+}$ and $\mathbf{E}_{n0}$ are defined as the following polynomials in the eigenvectors, for eigenvalues $\lambda = C$ and $\lambda = 0$. The outer products (or 'dyadic' (I3)), $\mathbf{X}_{n+}\mathbf{X}^{n+}$ and $\mathbf{X}_{n0}\mathbf{X}^{n0}$ on the right of these definitions, are $n \times n$ matrices, identical to the residual matrices (footnote [49]).

(C1) $\mathbf{E}_{n+} = (\mathbf{A}_{n0}^2 + C\mathbf{A}_{n0}) = \mathbf{X}_{n+}\mathbf{X}^{n+}, \ \lambda = C$

(C2) $\mathbf{E}_{n0} = (\mathbf{A}_{n0}^2 - C^2\mathbf{I}_n) = -\mathbf{X}_{n0}\mathbf{X}^{n0}, \ \lambda = 0$, see note (C18)

(C3) $$\mathbf{A}_{n0} = \begin{pmatrix} 0 & -t_n\mathbf{X}^{(n-1)-} \\ t_n\mathbf{X}_{(n-1)+} & \mathbf{A}_{(n-1)0} \end{pmatrix}.$$

Using $\mathbf{A}_{n0}$ the residual matrix $\mathbf{E}_{n+}$ is calculated as

(C4) $$\mathbf{E}_{n+} = \begin{pmatrix} 0 & 0 \\ 2t_nC\mathbf{X}_{(n-1)+} & -t_n^2\mathbf{X}_{(n-1)+}\mathbf{X}^{(n-1)-} + \mathbf{E}_{(n-1)+} \end{pmatrix}.$$

Given that $\mathbf{E}_{(n-1)+}$ and $\mathbf{X}_{n+}$ are defined as

(C5) $\mathbf{E}_{(n-1)+} = \mathbf{X}_{(n-1)+}\mathbf{X}^{(n-1)+}$.

(C6) $$\mathbf{X}_{n+} = \begin{pmatrix} 0 \\ \mathbf{X}_{(n-1)+} \end{pmatrix}, = \ldots \begin{pmatrix} 0 \\ \mathbf{X}_{3+} \end{pmatrix},$$

then, from $\mathbf{E}_{n+} = \mathbf{X}_{n+}\mathbf{X}^{n+}$ (C1), the vector $\mathbf{X}^{n+}$ is deduced to be

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47 The name 'residual' is unique to URMT and coined only for want of a name given none can be found in the literature, see [2] for some background.

48 The reciprocal is also known as the dual conjugate in URMT. Note that the dual conjugate is not the same as the transpose conjugate in URMT, see [1],#5.

49 The general residual matrix, as defined by an outer-product of vectors, e.g. $\mathbf{E}_{n+} = \mathbf{X}_{n+}\mathbf{X}^{n+}$, is actually a form of projection operator. Such operators are usually discussed under the subject of 'Spectral Decomposition' or 'Spectral Resolution', in linear algebra texts, see [5].

(C7) $\mathbf{X}^{n+} = -t_n^2\begin{pmatrix}0 & \mathbf{X}^{(n-1)-}\end{pmatrix} + 2t_n\begin{pmatrix}C & 0\end{pmatrix} + \begin{pmatrix}0 & \mathbf{X}^{(n-1)+}\end{pmatrix}.$

Using $\mathbf{A}_{n0}$ the residual matrix $\mathbf{E}_{n0}$ (C2) is calculated as

(C8) $\mathbf{E}_{n0} = \begin{pmatrix} -C^2 & t_nC\mathbf{X}^{(n-1)-} \\ t_nC\mathbf{X}_{(n-1)+} & -t_n^2\mathbf{X}_{(n-1)+}\mathbf{X}^{(n-1)-} + \mathbf{E}_{(n-1)0} \end{pmatrix}.$

The matrix $\mathbf{E}_{n0}$ is now split into two components, $\mathbf{E}_{n0}(n)$ and $\mathbf{E}_{n0}(n-1)$,

(C9) $\mathbf{E}_{n0}(n) = \begin{pmatrix} -C^2 & t_nC\mathbf{X}^{(n-1)-} \\ t_nC\mathbf{X}_{(n-1)+} & -t_n^2\mathbf{X}_{(n-1)+}\mathbf{X}^{(n-1)-} \end{pmatrix},$

(C10) $\mathbf{E}_{n0}(n-1) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{E}_{(n-1)0} \end{pmatrix}.$

The second component $\mathbf{E}_{n0}(n-1)$ is recursively defined in terms of $\mathbf{E}_{(n-1)0}$, which can be calculated using $\mathbf{E}_{(n-1)0} = (\mathbf{A}_{(n-1)0}^2 - C^2\mathbf{I}_{n-1}) = -\mathbf{X}_{(n-1)0}\mathbf{X}^{(n-1)0}$ etc.

Armed with the knowledge that $\mathbf{X}^{n0} = \left(\mathbf{T}^n\mathbf{X}_{n0}\right)^T$ (E3), and $\mathbf{E}_{n0} = -\mathbf{X}_{n0}\mathbf{X}^{n0}$ (C2), then $\mathbf{X}_{n0}$ and $\mathbf{X}^{n0}$ are

(C11) $\mathbf{X}_{n0} = -t_n\begin{pmatrix} 0 \\ \mathbf{X}_{(n-1)+} \end{pmatrix} + \begin{pmatrix} C \\ \mathbf{0}_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{X}_{(n-1)0} \end{pmatrix}$

(C12) $\mathbf{X}^{n0} = -t_n\begin{pmatrix}0 & \mathbf{X}^{(n-1)-}\end{pmatrix} + \begin{pmatrix}C & \mathbf{0}^{n-1}\end{pmatrix} + \begin{pmatrix}0 & \mathbf{X}^{(n-1)0}\end{pmatrix}.$

Using $\mathbf{X}^{n+}$ (C7), $\mathbf{X}_{n-}$ is then obtained from the $\mathbf{T}$ operator relation (E4)

(C13) $\mathbf{X}_{n-} = -t_n^2\begin{pmatrix} 0 \\ \mathbf{X}_{(n-1)+} \end{pmatrix} + 2t_n\begin{pmatrix} C \\ \mathbf{0}_{n-1} \end{pmatrix} + \begin{pmatrix} 0 \\ \mathbf{X}_{(n-1)-} \end{pmatrix}.$

Using $\mathbf{X}_{n+}$ (C6), $\mathbf{X}^{n-}$ is then obtained from the $\mathbf{T}$ operator relation (E2)

(C14) $\mathbf{X}^{n-} = \begin{pmatrix}0 & \mathbf{X}^{(n-1)-}\end{pmatrix}, = \ldots\begin{pmatrix}0 & \mathbf{X}^{3-}\end{pmatrix}$

For future reference, the two vectors $\mathbf{X}_{n-}$ (C13) and $\mathbf{X}^{n+}$ (C7) are usefully written in terms of the scale factors α_n, β_n and γ_n as

(C15) $$\mathbf{X}_{n-} = \begin{pmatrix} - \\ \alpha_n \\ \beta_n \\ -\gamma_n \end{pmatrix}, \ \mathbf{X}^{n+} = \begin{pmatrix} - & \alpha_n & \beta_n & \gamma_n \end{pmatrix}, \ n \geq 4$$

footnote[50]

and the scale factors are defined recursively as

(C16)

$$\alpha_n = -t_n^2 x + \alpha_{n-1}, \ n \geq 4$$
$$\beta_n = -t_n^2 y + \beta_{n-1}$$
$$\gamma_n = t_n^2 z + \gamma_{n-1}.$$

Some additional points:

(C17). The residual matrix method is the preferred method in URMT to evaluate eigenvectors since it automatically fixes the eigenvector scaling. Normally, eigenvectors are arbitrary to within a scale factor, but the residual method forces a fixed scale factor on a vector and its dual conjugate ([1],#5), e.g. $\mathbf{X}_{n+}$ and $\mathbf{X}^{n+}$. By imposing a primality (gcd) condition, e.g. (3.4c) for URM5, on the standard form of vector $\mathbf{X}_{n+}$ then the reciprocal vector $\mathbf{X}^{n+}$ is forced to take on whatever factor is needed to make $\mathbf{E}_{n+} = \mathbf{X}_{n+}\mathbf{X}^{n+}$. This can make $\mathbf{X}^{n+}$ non-primitive, but this is of no detrimental consequence - it is purely a scale factor and, as mentioned, perfectly legitimate for eigenvectors, which are arbitrary to within a scale factor.

(C18). There is a caveat to point (C17), which is that URM3 uses the standard $\mathbf{T}$ operator relation $\mathbf{X}^{30} = \left(\mathbf{T}^3\mathbf{X}_{30}\right)^T$ (A9c) that gives a sign for $\mathbf{X}^{30}$, opposite to that which would normally be obtained using a residual matrix $\mathbf{E}_{30}$, defined as $\mathbf{E}_{30} = \mathbf{X}_{30}\mathbf{X}^{30}$. This is intentional, to make the inner product $\mathbf{X}^{30}\mathbf{X}_{30} = +C^2$, and not $-C^2$, as this inner product represents the DCE (F3). Furthermore, using the standard form of the $\mathbf{T}$ operator relation $\mathbf{X}^{30} = \left(\mathbf{T}^3\mathbf{X}_{30}\right)^T$, to obtain $\mathbf{X}^{30}$ from $\mathbf{X}_{30}$, makes its derivation consistent with all other eigenvectors derived using $\mathbf{T}$ operator relations, Appendix (E).

(C19). The $\mathbf{E}_-$ residual matrix is not required here since $\mathbf{X}_{n-}$ and $\mathbf{X}^{n-}$ are obtained from the $\mathbf{T}$ operator relations (E2) and (E4). Given $\mathbf{X}_{n+}$ is pre-defined (13.2a), $\mathbf{X}^{n-}$ is obtained without any recourse to the residual method, using the $\mathbf{T}$ operator relation (E2). On the other

[50] The first blanked '-' element in $\mathbf{X}_{n-}$ is given by the summation term in (13.2b) involving matrix $\mathbf{M}_{k[n-(j+2)]}$, for $j = 1 \ldots n-3$.

hand, $\mathbf{X}_{n-}$ is obtained from the $\mathbf{T}$ operator relation (E4), using $\mathbf{X}^{n+}$ (C7), itself obtained from residual matrix $\mathbf{E}_{n+}$ (C4).

(C21). The quadratic, polynomial form, (C1) and (C2), for the residual matrices, $\mathbf{E}_{n+}$ and $\mathbf{E}_{n0}$ respectively, is the same for all URMn, $n \geq 3$, and determined by carefully selecting the conditions such that the eigenvalues are always the same two, non-zero values $\pm C$, with all the other eigenvalues zero.

Appendix (D)

22 Appendix (D) URM5 Example

This numeric example illustrates the compactification behaviour of the fifth dimension of URM5 as its evolution progresses.

In this example the evolutionary time t_5 of the fifth dimension in URM5 is varied from zero upward, whilst leaving all other times, t_3 and t_4, at their initial, zero value, in accordance with (10.4), and done primarily for simplicity, i.e. ease of understanding. In other words, there is no evolution in the first four dimensions, only the fifth.

(D1) $t_3 = 0$, $t_4 = 0$.

The example uses (embeds) the standard URM3 Pythagorean (4,3,5) solution, for unity eigenvalue, as given in the first row of table (A7) in Appendix (A). The solution is reproduced below.

(D2) $C = 1$, unity eigenvalue.
(C2) $x = 4$, $y = 3$, $z = 5$.
(C11a) $P = -1$, $Q = +2$, $R = -2$
(C11b) $\alpha = 0$, $\beta = -1$, $\gamma = 1$.

Using these values, the URM3 eigenvector solution is, by (A8), thus

(D3) $$\mathbf{X}_{3+} = \begin{pmatrix} 4 \\ 3 \\ 5 \end{pmatrix}, \ \mathbf{X}'_{30} = \begin{pmatrix} -1 \\ -2 \\ -2 \end{pmatrix}, \ \mathbf{X}'_{3-} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}.$$

These three URM3 eigenvectors are all static by virtue of evolutionary time t_3 constrained to zero (D1). Hence they all remain at their initial value and are superscripted with a prime to denote this, excepting $\mathbf{X}_{3+}$, which is always static and the prime omitted by assumption.

The URM5 eigenvector $\mathbf{X}_{5+}$ (11.1a) is also static, and remains at its initial, $\mathbf{X}_{3+}$ value, i.e.

(11.1a) $$\mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix}.$$

Because $t_3 = 0$ (D1) then $\mathbf{X}_{30} = \mathbf{X}'_{30}$ as a consequence and the URM5 eigenvector $\mathbf{X}_{50A}$ (11.1c) is also therefore static, remaining at its initial $\mathbf{X}'_{30}$ value, i.e.

$$\text{(D4)} \qquad \mathbf{X}_{50A} = \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}'_{30} \end{pmatrix}.$$

Lastly, on the subject of time-independent, static vectors, because $\mathbf{X}_{50B}$ is a function only of evolutionary time t_4, and since $t_4 = 0$ (D1), then $\mathbf{X}_{50B}$ reduces to the following static vector

$$\text{(D5)} \qquad \mathbf{X}_{50B} = \begin{pmatrix} 0 \\ C \\ \mathbf{0}_3 \end{pmatrix}, \; t_4 = 0.$$

Note that this vector $\mathbf{X}_{50B}$ has a constant component, eigenvalue C, in its fourth dimension; the eigenvalue being unity (D2) in this example. Ordinarily, C can be made as large as desired, see the comment (8.2b).

Using initial values $\mathbf{X}_{3+}$ and $\mathbf{X}'_{30}$ (D3), and $C = 1$ (D2), the static URM5 eigenvectors $\mathbf{X}_{5+}$, $\mathbf{X}_{50A}$ and $\mathbf{X}_{50B}$ are thus

$$\text{(D6)} \qquad \mathbf{X}_{5+} = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 3 \\ 5 \end{pmatrix}, \; \mathbf{X}_{50A} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ -2 \\ -2 \end{pmatrix}, \; \mathbf{X}_{50B} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; t_3 = 0, \; t_4 = 0.$$

This leaves just two, time-dependent URM5 eigenvectors $\mathbf{X}_{5-}$ (11.1b) and $\mathbf{X}_{50C}$ (11.1e), which, for $t_3 = 0$ and $t_4 = 0$ (D1), become

$$\text{(D7)} \qquad \mathbf{X}_{5-} = -t_5^2 \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + 2t_5 \begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}'_{3-} \end{pmatrix}.$$

$$\text{(D8)} \qquad \mathbf{X}_{50C} = -t_5 \begin{pmatrix} 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} C \\ 0 \\ \mathbf{0}_3 \end{pmatrix}.$$

Using initial values $\mathbf{X}_{3+}$ and $\mathbf{X}'_{3-}$ (D3), and $C=1$ (D2), then the initial values for the time-dependent vectors, all evolutionary times zero, are

(D9) $$\mathbf{X}'_{5-}=\begin{pmatrix}0\\0\\0\\-1\\-1\end{pmatrix},\ \mathbf{X}'_{50C}=\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix},\ t_3,t_4,t_5=0.$$

Calculations

The magnitude of the sum of two or more vectors is calculated from the sum of the individual magnitudes, which gives a maximum estimate of their size, i.e.

(D10) if $\mathbf{X}=\sum_i \mathbf{X}_i$ then $|\mathbf{X}|=\sum_i |\mathbf{X}_i|,\ i\geq 2$.

This does not apply to any individual eigenvector, whose magnitude is calculated as the positive, root sum of squares of its elements.

All magnitudes, except $|\mathbf{X}|_5$ (the magnitude of the fifth dimension), are calculated from the three-element vector of the first three dimensions, which correspond to the last three elements of the five-element, URM5 vectors, i.e. the URM3 subspace.

The magnitudes of the static URM3 vectors are

(D11)

(D11a) $|\mathbf{X}_{3+}|=5\sqrt{2}$, (D3)

(D11b) $|\mathbf{X}'_{30}|=3$, (D3)

(D11c) $|\mathbf{X}'_{3-}|=\sqrt{2}$, (D3).

The magnitudes of the static, URM5 vectors are as follows, **first three dimensions only**

(D12)

(D12a) $|\mathbf{X}_{5+}|=5\sqrt{2}$, (D6)

(D12b) $|\mathbf{X}_{50A}|=3$, (D6)

(D12c) $|\mathbf{X}_{50B}|=0$, (D6).

The two magnitudes, $\left|\mathbf{X}_{5-}\right|$ and $\left|\mathbf{X}_{50C}\right|$, **first three dimensions** only, are time-dependent, as follows

(D13)

(D13a) $\left|\mathbf{X}_{5-}\right| = \left|-t_5^2\mathbf{X}_{3+} + \mathbf{X}'_{3-}\right|$, exact

(D13b) $\left|\mathbf{X}_{5-}\right| = \sqrt{\left(50t_5^4 + 16t_5^2 + 2\right)}$, exact, from (D7) using (D3)

(D13c) $\left|\mathbf{X}_{5-}\right| \approx t_5^2\left|\mathbf{X}_{3+}\right|$, $t_5 >> 0$, large t_5 approximation from (D13a)

(D13d) $\left|\mathbf{X}_{5-}\right| \approx 5\sqrt{2}t_5^2$, $t_5 >> 0$, using (D11a)

(D13e) $\left|\mathbf{X}_{50C}\right| = t_5\left|\mathbf{X}_{3+}\right|$, exact, from (D8)

(D13f) $\left|\mathbf{X}_{50C}\right| = t_5 5\sqrt{2}$, using (D11a).

The magnitude of the fifth dimension $\left|\mathbf{X}\right|_5$ (6.1a)

(D14)

(D14a) $\left|\mathbf{X}\right|_5 = \sqrt{(2t_5C)^2 + C^2}$, exact, URM4 equivalent (9.1)

(D14b) $\left|\mathbf{X}\right|_5 = \sqrt{4t_5^2 + 1}$, $C = 1$ (D2)

(D14c) $\left|\mathbf{X}\right|_5 \approx 2t_5$, URM4 equivalent (9.3)

The magnitude of the first three dimensions of URM5 $\left|\mathbf{X}\right|_3$ (6.1b),

(D15)

(D15a) $\left|\mathbf{X}\right|_3 = \left|\mathbf{X}_{5+}\right| + \left|\mathbf{X}_{50A}\right| + \left|\mathbf{X}_{50B}\right| + \left|\mathbf{X}_{50C}\right| + \left|\mathbf{X}_{5-}\right|$, definition, (6.1b)

Defining constants a and b

(D15b) $a = 5\sqrt{2} = 7.071068$ to 6dps.

(D15c) $b = a + 3 = 10.071068$ to 6dps.

using (D12) and (D13) for the individual, URM5 vector magnitudes gives

(D15d) $\left|\mathbf{X}\right|_3 = b + at_5 + \left|\mathbf{X}_{5-}\right|$, exact, use (D13b) for $\left|\mathbf{X}_{5-}\right|$

Compactification Ratio

The true compactification ratio χ_5 (6.3)

(D16a) $\chi_5 = \frac{|\mathbf{X}|_5}{|\mathbf{X}|_3}$, definition (6.3)

(D16d) $\chi_5 = \frac{\sqrt{4t_5^2 + 1}}{b + at_5 + |\mathbf{X}_{5-}|}$, exact, using (D14b) and (D15d).

The approximated compactification ratio χ_5 (6.7)

(D17a) $\chi_5 \approx \frac{2C}{t_5|\mathbf{X}_{3+}|}$, (6.7), $t_5 \neq 0$: use (D16a) for $t_5 = 0$.

(D17b) $\chi_5 \approx \frac{\sqrt{2}}{5t_5}$, using (D11a), $t_5 \neq 0$ - see (D17a).

It is confirmed in both (D16d) and (D17b) that the compactification ratio limits to zero as t_5 grows to infinity since, for large t_5, both expressions are inversely proportional to t_5. Thus, the fifth dimension shrinks to zero relative to the first three dimensions, i.e.

(D18) $\lim_{t_5 \to \infty} \chi_5 = 0$ (6.4).

Error analysis

The absolute error ε_- (10.6)

(D19a) $\varepsilon_- = |\mathbf{X}_{5-}| - |t_5^2 \mathbf{X}_{3+}|$, definition, (10.6)

(D19b) $\varepsilon_- = |\mathbf{X}_{5-}| - 5\sqrt{2}t_5^2$, exact, using (D11a)

Expanding (D13b) $|\mathbf{X}_{5-}| = \sqrt{(50t_5^4 + 16t_5^2 + 2)}$ binomially, to first order in $1/t_5$, gives

(D19c) $|\mathbf{X}_{5-}| = 5\sqrt{2}t_5^2 + 4\sqrt{2}/5 + O(1/t_5^2)$ (footnote [51]).

[51] This approximation was not made in the main body of the paper, Section (10).

Substituting in this expansion (D19c) into (D19b) approximates the absolute error ε_{-} to the following constant, to first order in $1/t_5$,

(D19d) $$\varepsilon_{-} \approx \frac{4\sqrt{2}}{5} \approx 1.131371 \text{ to 6dps.}$$

Since the absolute error ε_{-} (D19d) is constant, it will shrink rapidly into insignificance compared with the quadratic term $5\sqrt{2}t_5^2$.

The relative error estimate ε (10.2)

(D20a) $$\varepsilon = \frac{\left\| \mathbf{X} \right|_3 - \left| t_5^2 \mathbf{X}_{3+} \right\|}{\left| \mathbf{X} \right|_3}, \text{ (10.2)}$$

Using (D15d) for $\left|\mathbf{X}\right|_3$ and definition (D19a) for the absolute error ε_{-}, the relative error estimate ε (D20a) becomes

(D20b) $$\varepsilon = \frac{b + at_5 + \varepsilon_{-}}{b + at_5 + at_5^2}.$$

From the approximation (D19d) for ε_{-}, the numerator is, to the highest order, linear in evolutionary time t_5. Hence, with a quadratic term at_5^2 in the denominator, it is confirmed from (D20b) that the error estimate ε limits to zero as t_5 grows to infinity, i.e.

(D20c) $$\lim_{t_5 \to \infty} \varepsilon = 0, \text{ (10.16)}$$

(D21) Tabulated Data

The following four sets of data are provided on the next few pages:

(D22) Time-dependent Vector $\mathbf{X}_{5-}$
(D23) Time-dependent Vector $\mathbf{X}_{50C}$
(D24) Time-dependent Magnitudes $\left|\mathbf{X}_{5-}\right|$, $\left|\mathbf{X}_{50C}\right|$, $\left|\mathbf{X}\right|_5$, $\left|\mathbf{X}\right|_3$
(D25) Compactification Ratios and Errors χ_5, ε, ε_-.

The column headings are defined as follows:

`t5` : evolutionary time t_5
`X-(1)` to `X-(5)` : five elements of time-dependent vector $\mathbf{X}_{5-}$ (D7)
`|X-|` : time-dependent magnitude $\left|\mathbf{X}_{5-}\right|$ (D13b), **first three dimensions** only
`X0C(1)` to `X0C(5)` : five elements of time-dependent vector $\mathbf{X}_{50C}$ (D8)
`|X0C|` : time-dependent magnitude $\left|\mathbf{X}_{50C}\right|$ (D13f), **first three dimensions** only
`|X|5` : The magnitude of the fifth dimension $\left|\mathbf{X}\right|_5$ (6.1a)
`|X|3` : The magnitude of the first three dimensions $\left|\mathbf{X}\right|_3$ (6.1b),
`chi5` : true compactification ratio χ_5 (D16a)
`chi5 app` : approximated compactification ratio χ_5 (D17b)
`chi5%err` : percentage error in χ_5 approximation, i.e. 100*(true-approx)/true
`eps` : relative error estimate ε (D20b)
`eps-` : absolute error ε_- (D19b)

(D22) Time-dependent Vector $\mathbf{X}_{5-}$

t5	X-(1)	X-(2)	X-(3)	X-(4)	X-(5)	|X-|
0,	0,	0,	0,	-1,	-1,	1.4,
1,	2,	0,	-4,	-4,	-6,	8.2,
2,	4,	0,	-16,	-13,	-21,	29.4,
3,	6,	0,	-36,	-28,	-46,	64.8,
4,	8,	0,	-64,	-49,	-81,	114.3,
5,	10,	0,	-100,	-76,	-126,	177.9,
6,	12,	0,	-144,	-109,	-181,	255.7,
7,	14,	0,	-196,	-148,	-246,	347.6,
8,	16,	0,	-256,	-193,	-321,	453.7,
9,	18,	0,	-324,	-244,	-406,	573.9,
10,	20,	0,	-400,	-301,	-501,	708.2,
11,	22,	0,	-484,	-364,	-606,	856.7,
12,	24,	0,	-576,	-433,	-721,	1019.4,
13,	26,	0,	-676,	-508,	-846,	1196.1,
14,	28,	0,	-784,	-589,	-981,	1387.1,
15,	30,	0,	-900,	-676,	-1126,	1592.1,
16,	32,	0,	-1024,	-769,	-1281,	1811.3,
24,	48,	0,	-2304,	-1729,	-2881,	4074.1,
32,	64,	0,	-4096,	-3073,	-5121,	7241.9,
48,	96,	0,	-9216,	-6913,	-11521,	16292.9,
64,	128,	0,	-16384,	-12289,	-20481,	28964.2,

(D23) Time-dependent Vector $\mathbf{X}_{50C}$

t5	X0C(1)	X0C(2)	X0C(3)	X0C(4)	X0C(5)	|X0C|
0,	1,	0,	0,	0,	0,	0.0,
1,	1,	0,	-4,	-3,	-5,	7.1,
2,	1,	0,	-8,	-6,	-10,	14.1,
3,	1,	0,	-12,	-9,	-15,	21.2,
4,	1,	0,	-16,	-12,	-20,	28.3,
5,	1,	0,	-20,	-15,	-25,	35.4,
6,	1,	0,	-24,	-18,	-30,	42.4,
7,	1,	0,	-28,	-21,	-35,	49.5,
8,	1,	0,	-32,	-24,	-40,	56.6,
9,	1,	0,	-36,	-27,	-45,	63.6,
10,	1,	0,	-40,	-30,	-50,	70.7,
11,	1,	0,	-44,	-33,	-55,	77.8,
12,	1,	0,	-48,	-36,	-60,	84.9,
13,	1,	0,	-52,	-39,	-65,	91.9,
14,	1,	0,	-56,	-42,	-70,	99.0,
15,	1,	0,	-60,	-45,	-75,	106.1,
16,	1,	0,	-64,	-48,	-80,	113.1,
24,	1,	0,	-96,	-72,	-120,	169.7,
32,	1,	0,	-128,	-96,	-160,	226.3,
48,	1,	0,	-192,	-144,	-240,	339.4,
64,	1,	0,	-256,	-192,	-320,	452.5,

(D24) Time-dependent Magnitudes $\left|\mathbf{X}_{5-}\right|$, $\left|\mathbf{X}_{50C}\right|$, $\left|\mathbf{X}\right|_5$, $\left|\mathbf{X}\right|_3$

t5	|X-|	|X0C|	|X|5	|X|3
0,	1.4,	0.0,	1.0,	11.5,
1,	8.2,	7.1,	2.2,	25.4,
2,	29.4,	14.1,	4.1,	53.6,
3,	64.8,	21.2,	6.1,	96.1,
4,	114.3,	28.3,	8.1,	152.6,
5,	177.9,	35.4,	10.0,	223.3,
6,	255.7,	42.4,	12.0,	308.2,
7,	347.6,	49.5,	14.0,	407.2,
8,	453.7,	56.6,	16.0,	520.3,
9,	573.9,	63.6,	18.0,	647.6,
10,	708.2,	70.7,	20.0,	789.0,
11,	856.7,	77.8,	22.0,	944.6,
12,	1019.4,	84.9,	24.0,	1114.3,
13,	1196.1,	91.9,	26.0,	1298.1,
14,	1387.1,	99.0,	28.0,	1496.1,
15,	1592.1,	106.1,	30.0,	1708.3,
16,	1811.3,	113.1,	32.0,	1934.5,
24,	4074.1,	169.7,	48.0,	4253.8,
32,	7241.9,	226.3,	64.0,	7478.3,
48,	16292.9,	339.4,	96.0,	16642.4,
64,	28964.2,	452.5,	128.0,	29426.8,

(D25) Compactification Ratios and Errors χ_5, ε, ε_-

t5,	chi5,	chi5 app,	chi5%err,	eps,	eps-,
0,	0.087,	0.000,	100.000,	0.000,	-----,
1,	0.088,	0.283,	-221.140,	0.721,	1.175,
2,	0.077,	0.141,	-83.987,	0.473,	1.144,
3,	0.063,	0.094,	-48.891,	0.338,	1.137,
4,	0.053,	0.071,	-33.863,	0.259,	1.135,
5,	0.045,	0.057,	-25.711,	0.208,	1.133,
6,	0.039,	0.047,	-20.650,	0.174,	1.133,
7,	0.034,	0.040,	-17.221,	0.149,	1.132,
8,	0.031,	0.035,	-14.752,	0.130,	1.132,
9,	0.028,	0.031,	-12.893,	0.116,	1.132,
10,	0.025,	0.028,	-11.445,	0.104,	1.132,
11,	0.023,	0.026,	-10.286,	0.094,	1.132,
12,	0.022,	0.024,	-9.339,	0.086,	1.132,
13,	0.020,	0.022,	-8.550,	0.079,	1.132,
14,	0.019,	0.020,	-7.882,	0.074,	1.132,
15,	0.018,	0.019,	-7.311,	0.069,	1.132,
16,	0.017,	0.018,	-6.817,	0.064,	1.132,
24,	0.011,	0.012,	-4.419,	0.043,	1.131,
32,	0.009,	0.009,	-3.267,	0.032,	1.131,
48,	0.006,	0.006,	-2.147,	0.021,	1.131,
64,	0.004,	0.004,	-1.598,	0.016,	1.131,

23 Appendix (E) Reciprocal Eigenvectors

The reciprocal (row-vector) forms of the standard, column eigenvectors are obtained using the appropriate dimensional $\mathbf{T}$ operator $\mathbf{T}_n$ $(\sim\mathbf{T}^n)$, $n \geq 2$ (this includes URM2)

(E1) $\mathbf{T}_n = \mathbf{T}^n = \begin{pmatrix} \mathbf{I}_{n-1} & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{I}_{n-1} = (n-1)\times(n-1)$ identity matrix

The $\mathbf{T}$ operator is of the same form as the familiar Minkowski metric of STR η_{ij}, disregarding sign convention. Within URMT the reciprocal eigenvectors[52], i.e. those with raised indices, e.g. $\mathbf{X}^{5-}$, are, indeed, related to the lowered index eigenvectors, $\mathbf{X}_{5+}$ in this example, via the relation $\mathbf{X}^{5-} = (\mathbf{T}^5\mathbf{X}_{5+})^T$ and, conversely, $\mathbf{X}_{5+} = (\mathbf{X}^{5-}\mathbf{T}_5)^T$.

Reciprocal eigenvectors for an n-dimensional vector space are only valid under Pythagoras conditions for that specific n-dimensional case, e.g. conditions (4.2) in the 5D case of URM5.

The reciprocal forms of the eigenvectors are not explicitly required in the paper for its central purpose, since all working can be done with the standard forms of eigenvectors. Nevertheless, most URMT conservation equations and scalar invariants arise from the inner products between the reciprocal and standard forms, as follows, see the general solution, Section (13) for the standard forms of the URMn vectors, from which the following reciprocal forms are obtained.

(E2) $\mathbf{X}^{n-} = (\mathbf{T}^n\mathbf{X}_{n+})^T$, $\mathbf{X}_{n+} = (\mathbf{X}^{n-}\mathbf{T}_n)^T$

(E3) $\mathbf{X}^{n0j} = (\mathbf{T}^n\mathbf{X}_{n0j})^T$, $j = 0\ldots n-1$,
where $\mathbf{X}^{n00} \sim \mathbf{X}^{n0A} \sim \mathbf{X}^{n0}$,
and $\mathbf{X}^{n01} \sim \mathbf{X}^{n0B}$, etc

(E4) $\mathbf{X}^{n+} = (\mathbf{T}^n\mathbf{X}_{n-})^T$, $\mathbf{X}_{n-} = (\mathbf{X}^{n+}\mathbf{T}_n)^T$

Examples

For URM5, the 5×5 matrix operator $\mathbf{T}_5$ $(=\mathbf{T}^5)$ is defined in block matrix form, using the 4×4 identity matrix $\mathbf{I}_4$

(E5) $\mathbf{T}_5 = \mathbf{T}^5 = \begin{pmatrix} \mathbf{I}_4 & 0 \\ 0 & -1 \end{pmatrix}$.

[52] Reciprocal eigenvectors are also known as conjugate vectors in [1] to [3]. Strictly speaking, they are the transpose conjugate, row-vector forms of the standard, column-vector forms, and vice versa. The concept of conjugacy within URMT covers all variables, matrices and vectors under a more unified approach given in [1],#5. Not least, it adds Hermitian-like properties to URMT, also desirable from a physical perspective.

Under Pythagoras conditions, there is a reciprocal vector $\mathbf{X}^{5-}$ defined in terms of $\mathbf{X}_{5+}$ by

(E6) $\mathbf{X}^{5-} = \left(\mathbf{T}^5 \mathbf{X}_{5+}\right)^T$.

Using the $\mathbf{T}^5$ operator (E5) and $\mathbf{X}_{5+}$ (5.2), then $\mathbf{X}^{5-}$ is thus,

(E7) $\mathbf{X}^{5-} = \begin{pmatrix} 0 & 0 & x & y & -z \end{pmatrix}$.

The Pythagoras equation (4.1) is now expressed as the following inner product of $\mathbf{X}^{5-}$ and $\mathbf{X}_{5+}$, which is a conservation equation, in URMT, with scalar invariant zero, see (F7),

(E8) $\mathbf{X}^{5-} \cdot \mathbf{X}_{5+} = x^2 + y^2 - z^2 = 0$.

The reciprocal form of the URM4 eigenvector solution (8.1) is given below in terms of the URM3 reciprocal vectors

(E9)

(E9a) $\mathbf{X}^{4-} = \begin{pmatrix} 0 & \mathbf{X}^{3-} \end{pmatrix}$

(E9b) $\mathbf{X}^{4+} = -t_4^2 \begin{pmatrix} 0 & \mathbf{X}^{3-} \end{pmatrix} + 2t_4 \begin{pmatrix} C & \mathbf{0}^3 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{X}^{3+} \end{pmatrix}$

(E9c) $\mathbf{X}^{40A} = \begin{pmatrix} 0 & \mathbf{X}^{30} \end{pmatrix}$

(E9d) $\mathbf{X}^{40B} = -t_4 \begin{pmatrix} 0 & \mathbf{X}^{3-} \end{pmatrix} + \begin{pmatrix} C & \mathbf{0}^3 \end{pmatrix}$

The reciprocal form of the URM5 eigenvector solution (11.1) is given below in terms of the URM3 reciprocal vectors

(E10)

(E10a)
$$\begin{aligned} &\mathbf{X}^{5+} = -(t_5^2 + t_4^2)\begin{pmatrix} 0 & 0 & \mathbf{X}^{3-} \end{pmatrix} + \\ &2t_5 \begin{pmatrix} C & 0 & \mathbf{0}^3 \end{pmatrix} + \\ &2t_4 \begin{pmatrix} 0 & C & \mathbf{0}^3 \end{pmatrix} + \\ &\begin{pmatrix} 0 & 0 & \mathbf{X}^{3+} \end{pmatrix} \end{aligned}$$

(E10b) $\mathbf{X}^{5-} = \begin{pmatrix} 0 & 0 & \mathbf{X}^{3-} \end{pmatrix}$

(E10c) $\mathbf{X}^{50A} = \begin{pmatrix} 0 & 0 & \mathbf{X}^{30} \end{pmatrix}$

(E10d) $\mathbf{X}^{50B} = -t_4 \begin{pmatrix} 0 & 0 & \mathbf{X}^{3-} \end{pmatrix} + \begin{pmatrix} 0 & C & \mathbf{0}^3 \end{pmatrix}$

(E10e) $\mathbf{X}^{50C} = -t_5 \begin{pmatrix} 0 & 0 & \mathbf{X}^{3-} \end{pmatrix} + \begin{pmatrix} C & 0 & \mathbf{0}^3 \end{pmatrix}$

24 Appendix (F) Conservation Equations and Scalar Invariants

Following on from Appendix (E), the six key conservation equations of URMT, as obtained from the inner product relations between the eigenvectors, are given in this Appendix for both URM3 and the general, n-dimensional case.

The URM3 conservation equations are:

(F1) $\mathbf{X}^{3-}\mathbf{X}_{3+} = x^2 + y^2 - z^2 = 0$ Pythagoras

(F2) $\mathbf{X}^{3+}\mathbf{X}_{3-} = \alpha_3^2 + \beta_3^2 - \gamma_3^2 = 0$ Pythagoras

(F3) $\mathbf{X}^{30}\mathbf{X}_{30} = P^2 + Q^2 - R^2 = +C^2$ the DCE

(F4) $\mathbf{X}^{3+}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{3-} = \alpha_3 x + \beta_3 y + \gamma_3 z = +2C^2$ the Potential Equation ($V = 0$)

(F5) $\mathbf{X}^{30}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{30} = xP - yQ - zR = 0$ the Delta equation

(F6) $\mathbf{X}^{30}\mathbf{X}_{3-} = \mathbf{X}^{3+}\mathbf{X}_{30} = \alpha_3 P - \beta_3 Q + \gamma_3 R = 0$ the Dual Delta equation

The n-dimensional forms of these equations, for URMn, $n \geq 4$ give exactly the same invariants. They are stated below, with an example given following of how they are proved by induction. Note that the scale factors α_n, β_n and γ_n are defined recursively in (C16).

(F7) $\mathbf{X}^{n-}\mathbf{X}_{n+} = 0$, Pythagoras

(F8) $\mathbf{X}^{n+}\mathbf{X}_{n-} = \sum_{j=4}^{n}(2t_j C)^2 + \alpha_n^2 + \beta_n^2 - \gamma_n^2 = 0$, Pythagoras

(F9) $\mathbf{X}^{n+}\mathbf{X}_{n+} = \mathbf{X}^{n-}\mathbf{X}_{n-} = \alpha_n x + \beta_n y + \gamma_n z = +2C^2$ the Potential Equation ($V = 0$)

Using a zero vector $\mathbf{X}_{n0}$ given by the sum $\mathbf{X}_{n0} = \sum_{j=0}^{n-1}\mathbf{X}_{n0j}$, where $\mathbf{X}_{n00} \sim \mathbf{X}_{n0A}$, $\mathbf{X}_{n01} \sim \mathbf{X}_{n0B}$, $\mathbf{X}_{n02} \sim \mathbf{X}_{n0C}$ etc., and related reciprocal vector $\mathbf{X}^{n0} = \left(\mathbf{T}^n \mathbf{X}_{n0}\right)^T$, then

(F10) $\mathbf{X}^{n0}\mathbf{X}_{n0} = P^2 + Q^2 - R^2 = (n-2)C^2$, the DCE

Note that the following orthogonality relation holds between different, zero eigenvectors:

(F10b) $\mathbf{X}^{n0j}\mathbf{X}_{n0i} = C^2$ if $i = j$, and $\mathbf{X}^{n0j}\mathbf{X}_{n0i} = 0$ if $i \neq j$.

(F11) $\mathbf{X}^{n0}\mathbf{X}_{n+} = \mathbf{X}^{n-}\mathbf{X}_{n0} = 0$ the Delta equation

(F12) $\mathbf{X}^{n0}\mathbf{X}_{n-} = \mathbf{X}^{n+}\mathbf{X}_{n0} = \alpha_n P - \beta_n Q + \gamma_n R = 0$ the Dual Delta equation

These relations can be proved inductively using a recursive form of the general solutions provided in Appendix (C), to give the eigenvector solutions for the n-dimensional case in terms of the $n-1$ dimensional eigenvectors.

An example of this is given following using the n-dimensional eigenvectors $\mathbf{X}_{n+}$ (13.2a) and $\mathbf{X}^{n+}$ (C7). Although this is not particularly pertinent to the paper's theme, it is considered useful as a good illustration in standard URMT algebra.

To prove the inner product relation (F9) inductively then, starting with their general solutions $\mathbf{X}_{n+}$ (13.2a) and $\mathbf{X}^{n+}$ (C7), reproduced below,

(F13) $$\mathbf{X}_{n+} = \begin{pmatrix} 0 \\ \mathbf{X}_{(n-1)+} \end{pmatrix} \text{ (13.2a)},$$

$$\mathbf{X}^{n+} = \begin{pmatrix} 2t_{n-1}C & -t_n^2\mathbf{X}^{(n-1)-} + \mathbf{X}^{(n-1)+} \end{pmatrix} \text{ (C7)},$$

the inner product $\mathbf{X}^{n+}\mathbf{X}_{n+}$ of the two is given by

(F14) $$\mathbf{X}^{n+}\mathbf{X}_{n+} = -t_n^2\mathbf{X}^{(n-1)-}\mathbf{X}_{(n-1)+} + \mathbf{X}^{(n-1)+}\mathbf{X}_{(n-1)+}.$$

The right of this product is given purely in terms of the $n-1$ dimensional eigenvectors, albeit there are now two terms.

If it can be shown that the first term $\mathbf{X}^{(n-1)-}\mathbf{X}_{(n-1)+}$ is zero[53], then the product will reduce to $\mathbf{X}^{n+}\mathbf{X}_{n+} = \mathbf{X}^{(n-1)+}\mathbf{X}_{(n-1)+}$, i.e. the desired reduction of $\mathbf{X}^{n+}\mathbf{X}_{n+}$ to $\mathbf{X}^{(n-1)+}\mathbf{X}_{(n-1)+}$ will have been achieved. Really, of course, the inductive argument shows that, if it is true for the $n-1$ case then it is true for the n case - this argument is given at the end.

It is relatively trivial to prove $\mathbf{X}^{(n-1)-}\mathbf{X}_{(n-1)+}$ is zero because the vector $\mathbf{X}_{(n-1)+}$ is static, as seen by the recursive formula for $\mathbf{X}_{n+}$

(F15) $$\mathbf{X}_{n+} = \mathbf{X}_{(n-1)+} = \mathbf{X}_{(n-2)+} = \ldots \begin{pmatrix} \mathbf{0}_{n-3} \\ \mathbf{X}_{3+} \end{pmatrix}.$$

Likewise, for $\mathbf{X}^{(n-1)-}$, since it is simply obtained from the relation $\mathbf{X}^{(n-1)-} = \left(\mathbf{T}^{n-1}\mathbf{X}_{(n-1)+}\right)^T$, i.e.

(F16) $$\mathbf{X}^{(n-1)-} = \begin{pmatrix} \mathbf{0}^{n-3} & \left(\mathbf{T}^3\mathbf{X}_{3+}\right)^T \end{pmatrix},$$

and, using the URM3 relation $\left(\mathbf{T}^3\mathbf{X}_{3+}\right)^T = \mathbf{X}^{3-}$, then $\mathbf{X}^{(n-1)-}$ is given in terms of URM3 vectors as

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[53] Since $\mathbf{X}^{(n-1)-}$ is a reciprocal eigenvector for eigenvalue $-C$, and $\mathbf{X}_{(n-1)+}$ is a standard eigenvector for eigenvalue $+C$ then, by the rules of matrix algebra, these eigenvectors will be orthogonal, i.e. their inner product zero. This is usually described in the literature [5] under the subject of 'orthogonality' of eigenvectors to different eigenvalues. Note that this orthogonality is between reciprocal (row-vector) and standard (column-vector) forms, but not between standard-standard or reciprocal-reciprocal vector forms. In these two latter cases, as noted in URMT (footnote 37), the standard vectors form a highly oblique basis, non-orthogonal basis and so too, therefore, the reciprocal vectors.

(F17) $\mathbf{X}^{(n-1)-} = \begin{pmatrix}\mathbf{0}^{n-3} & \mathbf{X}^{3-}\end{pmatrix}$.

Thus, using this and the results for $\mathbf{X}^{(n-1)-}$ (F16) and $\mathbf{X}_{(n-1)+}$ (F15), the inner product $\mathbf{X}^{(n-1)-}\mathbf{X}_{(n-1)+}$ is given in terms of URM3 vectors as

(F18) $\mathbf{X}^{(n-1)-}\mathbf{X}_{(n-1)+} = \mathbf{X}^{3-}\mathbf{X}_{3+}$.

Since $\mathbf{X}^{3-}\mathbf{X}_{3+}$ is zero by Pythagoras (F1), then it is proved that the first term on the right of (F14) is zero, i.e.

(F19) $\mathbf{X}^{(n-1)-}\mathbf{X}_{(n-1)+} = 0$,

and the inner product (F14) becomes

(F20) $\mathbf{X}^{n+}\mathbf{X}_{n+} = \mathbf{X}^{(n-1)+}\mathbf{X}_{(n-1)+}$.

Retracing, if $\mathbf{X}^{(n-1)+}\mathbf{X}_{(n-1)+} = +2C^2$, then so too $\mathbf{X}^{n+}\mathbf{X}_{n+} = +2C^2$ by (F20). Since it is true for $n = 3$, i.e. $\mathbf{X}^{3+}\mathbf{X}_{3+} = +2C^2$ (F4), then it is also therefore true for $n = 4,5\ldots$ etc, hence (F9) is proven for all $n \geq 3$.

(F21) **Commentary**

The n-dimensional vector space is characterised by n, independent parameters (k, l and t_j, $j = 3\ldots n$), all but two (k and l, see footnote (7)) are physically interpreted as temporal coordinates. Each jth dimension, three and higher, has evolutionary behaviour governed by its jth temporal coordinate t_j, effectively making the eigenvector space a discrete set of, n-dimensional points, termed the 'lattice' in [1],#3 and [3]. A point in the lattice is therefore uniquely specified by the n-element, coordinate vector $\begin{pmatrix}k & l & t_j\end{pmatrix}$, $j = 3\ldots n$. Every lattice point is characterised by a set of invariants $0,\pm C^2,\pm 2C^2$ (footnote [54]), given by the scalar products, (F7) to (F12), between the eigenvectors. For the unity eigenvalue, $C = 1$, this gives the set $0,\pm 1,\pm 2$. Ratios of these (except zero) may also be considered. Regardless of the size of the eigenvectors, and their respective elements, which could easily be $O(10^1)$ to $O(10^{80})$, the same three numbers, 0, 1 and 2 appear at every lattice position. What do these integer invariants represent? Are their ratios meaningful?

[54] The minus sign can be selected using a different sign convention for the eigenvectors.

25 Appendix (G) Calculus Properties of URMT

Whilst URM3 vectors $\mathbf{X}_{3+}$, $\mathbf{X}_{30}$, $\mathbf{X}_{3-}$ can be consistently interpreted in terms of their physical units, with an acceleration, velocity and position vector respectively, they are also related via the following calculus relations, further justifying the standard physical interpretation (2.1), see also [1],#3 and [3].

The standard calculus derivative $\frac{d}{dt_3}$ (~$\frac{d}{dm}$ in [1], [3]) is used as a good, large t_3 approximation for discrete differences (see also point (G17) below), i.e.

(G1) $\frac{d}{dt_3} \approx \frac{\delta}{\delta t_3}$, $t_3 >> 0$, $\delta t_3 = 1$,

(G2) $\frac{d\mathbf{X}_{3-}}{dt_3} = 2\mathbf{X}_{30}$, derivative of position = twice velocity

(G3) $\frac{d\mathbf{X}_{30}}{dt_3} = -\mathbf{X}_{+}$, derivative of velocity = negative of acceleration

(G4) $\frac{d\mathbf{X}_{3+}}{dt_3} = 0$, derivative of acceleration = zero (constant acceleration)

(G5) $\frac{d^2\mathbf{X}_{3-}}{d^2t_3} = -2\mathbf{X}_{3+}$, second derivative of position = - twice acceleration

Higher dimensional, extension work in [4] shows these identical relationships are maintained as follows for general, URMn.

Calculus Properties of URMn

With more than one evolutionary parameter for four and higher dimensions, the standard calculus partial derivative $\frac{\partial}{\partial t_i}$ is now used in place of $\frac{d}{dt_3}$ for derivatives with respect to evolutionary time t_i.

(G6) $\frac{\partial}{\partial t_i} \approx \frac{\delta}{\delta t_i}$, $t_i >> 0$, $\delta t_i = 1$,

For URM5, the partial derivatives are

(G7) $\frac{\partial\mathbf{X}_{5-}}{\partial t_4} = 2\mathbf{X}_{50B}$, derivative of position = twice velocity

(G8) $\dfrac{\partial \mathbf{X}_{5-}}{\partial t_5} = 2\mathbf{X}_{50C}$, ditto

(G9) $\dfrac{\partial \mathbf{X}_{50A}}{\partial t_3} = -\mathbf{X}_{5+}$, derivative of velocity = negative of acceleration

(G10) $\dfrac{\partial \mathbf{X}_{50B}}{\partial t_4} = -\mathbf{X}_{5+}$, ditto

(G11) $\dfrac{\partial \mathbf{X}_{50C}}{\partial t_5} = -\mathbf{X}_{5+}$, ditto

(G12) $\dfrac{\partial \mathbf{X}_{5+}}{\partial t_4} = 0$, $\dfrac{\partial \mathbf{X}_{5+}}{\partial t_5} = 0$, constant acceleration

(G13) $\dfrac{\partial^2 \mathbf{X}_{5-}}{\partial^2 t_4} = -2\mathbf{X}_{5+}$, $\dfrac{\partial^2 \mathbf{X}_{5-}}{\partial^2 t_5} = -2\mathbf{X}_{5+}$ second derivative of position = - twice acceleration

For URMn, dimension $j+3$, $j = 0 \ldots n-3$, ($j = 0$ includes URM3 here) the same general relations hold true for evolutionary parameter t_{j+3}, e.g. t_3 for $j = 0$, t_4 for $j = 1$, etc. and t_n for $j = n-3$.

(G14) $\dfrac{\partial \mathbf{X}_{n-}}{\partial t_{j+3}} = 2\mathbf{X}_{n0j}$, derivative of position = twice velocity

(G15) $\dfrac{\partial \mathbf{X}_{n0j}}{\partial t_{j+3}} = -\mathbf{X}_{n+}$, derivative of velocity = negative of acceleration

(G16) $\dfrac{\partial^2 \mathbf{X}_{n-}}{\partial^2 t_{j+3}} = -2\mathbf{X}_{n+}$, second derivative of position = - twice acceleration

Some additional points

(G17) These calculus relations sometimes have the caveat of valid only for large evolutionary times. This caveat is unrelated to any compactification issues, and is only required in so far as the continuous derivative is used as an approximation for the discrete difference for any large time t_j, $t_j >> 0$.

(G18) The n vectors can be physically associated with a single acceleration (the same for all dimensions), $n-2$ velocity vectors, and a single position vector, all related by standard, calculus relations. Except there is no calculus in URMT's formulation, only an invariance principle and associated invariance transformations, but absolutely no calculus or difference equations used.

26 Appendix (H) URM2

The key matrices and eigenvectors in the 2×2 formulation of URMT (URM2), when under URM2 Pythagoras conditions, are as follows, reproduced from [4]. URM2 is also a special case of what is termed 'the almost trivial' solution in Appendix C of [1],#3.

(H1) $\mathbf{A}_{20} = \begin{pmatrix} 0 & C \\ C & 0 \end{pmatrix}$, 'unity' root matrix

(H2) $\lambda^2 - C^2 = 0$, $\lambda \pm C$, eigenvalues

(H3) $\mathbf{X}_{2+} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\mathbf{X}_{2-} = \begin{pmatrix} C^2 \\ -C^2 \end{pmatrix}$, standard eigenvectors

(H4) $\mathbf{X}^{2+} = \begin{pmatrix} C^2 & C^2 \end{pmatrix}$, $\mathbf{X}^{2-} = \begin{pmatrix} 1 & -1 \end{pmatrix}$, reciprocal eigenvectors

(H5) $\mathbf{T}_2 = \mathbf{T}^2 = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$, the $\mathbf{T}$ operator

(H6) $K = C^2$, $V = 0$, kinetic and Potential terms

(H7) $C^2 = K + V$ (~ $C^2 = \lambda^2$), the DCE

In URM2, the total energy E is always the kinetic term K (H6), and the Potential V is always zero, with no pre-conditions such as the Pythagoras conditions, Section (4).

The only free parameter within URM2 is actually the eigenvalue C which, by definition, is unity or greater.

Generally, this URM2 case is considered too simplistic, primarily because the only non-trivial, primitive $\mathbf{X}_{2+}$ vector is the $(1,1)$ pair, which is why URMT generally starts with URM3 since it has the first 'non-trivial' solution $\mathbf{X}_{3+}$ - an arbitrary Pythagorean triple with three, non-zero elements.

Neither is their any meaningful variational (or evolutionary) parameter t_2, see [4]. But this is more of a plus point, because it means URM2 cannot shrink further from two to one dimension.

Note that these two aforementioned points might actually be telling us something about 3D?

Despite URM2 being considered too simplistic, it is not entirely dismissed and, if for no other reason, it is a good illustration of some basic aspects of URMT. Most importantly amongst these is that the above $(1,1)$ solution can be 'lifted' (I6) to a general, 3D Pythagorean solution, using a matrix $\mathbf{A}_{30}$ and eigenvector $\mathbf{X}_{3+}$, based on the general n-dimensional matrix $\mathbf{A}_{n0}$ and eigenvector $\mathbf{X}_{n+}$, Appendix (C). Using $\mathbf{A}_{n0}$ with $n=3$, the 3D matrix $\mathbf{A}_{30}$ and vector $\mathbf{X}_{3+}$ are defined as

(H8) $$\mathbf{A}_{30} = \begin{pmatrix} 0 & -t_3\mathbf{X}^{2-} \\ t_3\mathbf{X}_{2+} & \mathbf{A}_{20} \end{pmatrix},\ \mathbf{X}_{3+} = \begin{pmatrix} 0 \\ \mathbf{X}_{2+} \end{pmatrix}.$$

Substituting for matrix $\mathbf{A}_{20}$ (H1) and eigenvectors $\mathbf{X}_{2+}$ (H3), $\mathbf{X}^{2-}$ (H4), then $\mathbf{A}_{30}$ and $\mathbf{X}_{3+}$ are expanded in full as

(H9) $$\mathbf{A}_{30} = \begin{pmatrix} 0 & -t_3 & +t_3 \\ +t_3 & 0 & C \\ +t_3 & C & 0 \end{pmatrix},\ \mathbf{X}_{3+} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

To obtain $\mathbf{X}^{3+}$, the residual matrix method, Appendix (C), is used to calculate the residual matrix $\mathbf{E}_{3+}$

(H10) $$\mathbf{E}_{3+} = \mathbf{A}_{30}^2 + C\mathbf{A}_{30},$$

and then extracting $\mathbf{X}^{3+}$ using the equivalent definition $\mathbf{E}_{3+} = \mathbf{X}_{3+}\mathbf{X}^{3+}$, gives $\mathbf{X}^{3+}$ as follows

(H11) $$\mathbf{X}^{3+} = \begin{pmatrix} 2t_3C & C^2 - t_3^2 & C^2 + t_3^2 \end{pmatrix}.$$

It can be seen that the three elements of this vector are the standard parameterisation of a Pythagorean triple for arbitrary integer parameters t_3, C.

With $\mathbf{X}^{3+}$ defined in the usual way in terms of URM3 scale factors α_3, β_3 and γ_3, (A10a), then, comparing with (H11), they are thus

(H12) $$\alpha_3 = 2t_3C,\ \beta_3 = C^2 - t_3^2,\ \gamma_3 = C^2 + t_3^2$$

and the Pythagoras equation is simply

(H13) $$\gamma_3^2 = \alpha_3^2 + \beta_3^2.$$

With the 3D $\mathbf{T}$ operator, $\mathbf{T}_3$ ($\sim\mathbf{T}^3$), defined as

(H14) $$\mathbf{T}_3 = \mathbf{T}^3 = \begin{pmatrix} +1 & 0 \\ 0 & \mathbf{T}_2 \end{pmatrix},$$

then $\mathbf{X}_{3-}$ is obtained in the usual way $\mathbf{X}_{3-} = (\mathbf{T}_3\mathbf{X}^{3+})^T$

(H15) $\mathbf{X}_{3-} = \begin{pmatrix} 2t_3C \\ C^2 - t_3^2 \\ -\left(C^2 + t_3^2\right) \end{pmatrix}$

Lastly, the zero eigenvector $\mathbf{X}_{30}$ is

(H16) $\mathbf{X}_{30} = \begin{pmatrix} C \\ -t_3 \\ -t_3 \end{pmatrix}$

and its reciprocal $\mathbf{X}^{30}$ obtained from $\mathbf{X}^{30} = (\mathbf{T}^3\mathbf{X}_{30})^T$

(H17) $\mathbf{X}^{30} = \begin{pmatrix} C & -t_3 & +t_3 \end{pmatrix}$.

The DCE in scalar product form, (F3), is then verified to be the conserved quantity C^2, i.e.

(H18) $\mathbf{X}^{30} \cdot \mathbf{X}_{30} = C^2$.

This completes the overview of URM2.

27 Appendix (I) Terminology

This is a subset of the full URMT terminology, covering terms used herein, either uniquely defined within URMT or those in wider use, but with a specific meaning to URMT.

(I1) **Alignment** - see Flattening (I5).

(I2) **Divisibility factors**, also known as **scale factors**, are the last three elements of the URMn eigenvector $\mathbf{X}^{n+}$, denoted by α_n, β_n and γ_n, $n \geq 3$, see (A15) URM3 and (C16) UMRn.

(I3) The **Dyadic product** is synonymous with the outer product of two vectors. In the context of URMT, the dyadic product of two vectors $\mathbf{X}_i$ and $\mathbf{X}^j$, $i, j = 1 \ldots n$, gives a square matrix, $\mathbf{M}_{ij} = \mathbf{X}_i \mathbf{X}^j$, of size $n \times n$. The term 'dyadic product' is quite old and is replaced by 'outer product' in modern texts. Nevertheless, the name appears in the earlier URMT literature, e.g. [1], albeit it has been replaced herein with the term 'outer product'.

(I4) An **Excess dimension** is any dimension higher than the third, i.e. the fourth or higher. In an n-dimensional space, an excess dimension r is such that $3 < r < n$; its associated temporal (evolutionary) parameter is denoted by t_r, see Appendix (C).

(I5) **Flattening** is the term used to describe the eigenvector evolution in URM3, whereby the two eigenvectors $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ align anti-parallel to $\mathbf{X}_{3+}$ as evolution progresses, i.e. as evolutionary time t_3 increases, see Appendix (B) for the URM3 eigenvector evolution equations. Because it is an alignment of vectors, the process is also known as "alignment" herein, but 'flattening' is used exclusively in earlier, URM3 literature; see [1],#3 for full details of the evolutionary process, and [3] for a summary.

(I5) **Flattening** is the term used to describe the eigenvector evolution in URM3 whereby the two eigenvectors $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$ align anti-parallel to $\mathbf{X}_{3+}$ as evolution progresses, i.e. as evolutionary time m (or t_3) increases, see Appendix (B) for the URM3 eigenvector evolution equations. The vector $\mathbf{X}_{3+}$ itself is static and invariant to arbitrary variations in any evolutionary parameter. However, note that $\mathbf{X}_{3+}$ is actually a two-parameter family of integer vectors, parameters k and l (A26), and hence occupies a 2D discrete subspace of 3D; in this sense, the 3D flattens to 2D. Because it is an alignment of vectors, the process is also known as 'alignment' herein, but 'flattening' is used exclusively in earlier, URM3 literature. That they align anti-parallel, and not parallel, is largely a choice of sign convention. See [1],#3 for full details of the evolutionary process, and [3] for a summary.

(I6) **Lifting**, in the context of URMT, is the process of generating eigenvector solutions for an $(n+1) \times (n+1)$ matrix $\mathbf{A}_{n+1}$ using an eigenvector solution to the $n \times n$ matrix $\mathbf{A}_n$, $n \geq 2$. The matrix $\mathbf{A}_n$ is embedded in $\mathbf{A}_{n+1}$ and an eigenvector solution $\mathbf{X}$, to $\mathbf{A}_n$, is also a solution to $\mathbf{A}_{n+1}$, with appropriate zero padding.

(I7) A **'non-trivial' vector** is one with all non-zero elements and, in the context of URMT, remains non-trivial for arbitrary variations in all evolutionary parameters. Primarily this applies to the static vector $\mathbf{X}_{n+}$ and its reciprocal $\mathbf{X}^{n-}$, since $\mathbf{X}_{n+}$ is the only true, invariant vector.

(I8) A **primitive Pythagorean n-tuple** is that which has no common factor in its elements, as specified by gcd criterion, e.g. URM5 gcd criterion (3.4c). The vector $\mathbf{X}_{n+}$ and its reciprocal $\mathbf{X}^{n-}$ are usually defined to be primitive, with any common factor being absorbed into $\mathbf{X}_{n-}$ and its reciprocal $\mathbf{X}^{n+}$, which are also Pythagorean n-tuples.

(I9) The **Pythagoras conditions** are a set of relations between the standard and conjugate dynamical variables in URMn, which are such that the eigenvectors of the $n \times n$ matrix $\mathbf{A}_n$, for non-zero eigenvalues, are Pythagorean n-tuples. The unity root matrix $\mathbf{A}_n$ is formed exclusively from the dynamical variables, and the conditions make the matrix skew-symmetric in the first, $n-1$ rows and columns, and symmetric in the last row and column. All Pythagoras conditions for URMn include URM$(n-1)$ as a subset.

(I10) **Scale factors**, see divisibility factors.

(I11) A **Static** quantity in URMT is any quantity (invariably an eigenvector) not dependent on any evolutionary time t_j, $j = 3 \ldots n$. The eigenvector $\mathbf{X}_{n+}$, $n \geq 2$ (this includes URM2), is the classic URMT example. A static eigenvector can be a function of none, one or both of the other two URMT parameters, k and l, which are not temporal parameters, e.g. $\mathbf{X}_{n+}$ is a function of both k and l.

(I12) **Zero Eigenvectors.** The eigenvectors $\mathbf{X}_{30}$, $\mathbf{X}_{n0A}$, $\mathbf{X}_{n0B}$, $\mathbf{X}_{n0C}$ etc., are called zero eigenvectors since they are the eigenvectors for the repeated, zero eigenvalue and not because they have all elements zero.