Hypercharge Quantisation and Fermat's Last Theorem

Richard J. Miller
richard@microscitech.com,
http://www.urmt.org

Abstract

This paper shows how the quark Hypercharge values, in integer form, for the SU(3), Up, Down and Strange Quark flavour model, are given by the eigenvalues of a special eigenvector solution to a generalised form of Fermat's Last Theorem, which has integer solutions for arbitrary odd exponent greater than two. Upon reduction of the order of the equation to a quadratic exponent, and hence Pythagorean in form, the eigenvalues become those of the third component of Isospin. The associated eigenvectors representing the Up and Down quarks are seen to algebraically differentiate themselves from that of the Strange quark, and this is noted as indicative of the relative stability of nucleons formed from the Up and Down quarks compared with that of the Strange quark, which forms unstable particles.

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1 Introduction

A modified form of Fermat's Last Theorem (FLT), known as the 'coordinate equation', further below (1), was first published by the author in [1], together with numerous, speculative links to Physics. Later work focussed on a simplified, quadratic form of the coordinate equation culminating, after several years, with an integer representation of the Quark Flavour Model [2] and associated integer eigenvalues representing the various quark charges including Isospin and Hypercharge. Isospin eigenvalues arise from a simplified, quadratic solution to the Coordinate Equation, whilst Hypercharge eigenvalues naturally arise from the same equation, but without the restriction to a quadratic exponent and valid for all odd exponent solutions, cubic and higher. It is noted that a recent paper [3] also makes a link between Hypercharge and FLT for a cubic-only form, but from a different perspective to this paper.

The Coordinate Equation

The coordinate equation is given in terms of three integers variables x, y, z, for some non-zero integer k, as

$$0 = x^n + y^n - z^n + kxyz \tag{1}$$

The full list of conditions is

$$x, y, z, k, n \in \mathbb{Z},\ k \neq 0,\ n \geq 2,\ 1 < x < y < z$$

$$\gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1$$

Were k to be zero then this equation would be reduce to the famous equation of Fermat's Last Theorem (FLT) but, as proven by Wiles [4], there are no integer solutions for $k = 0$. Nevertheless, for certain non-zero k, there is an infinite set of integer solutions x, y, z for all exponents $n \geq 2$, and numerous example solutions are given in [5] for a variety of exponents together with the values for k.

By defining a vector $\mathbf{X}_+$ comprising such a solution to (1)

$$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{2}$$

then, in accordance with [1], this vector is an eigenvector of a 'unity root matrix' $\mathbf{A}$ (defined next), for unity eigenvalue, i.e.

$$\mathbf{A}\mathbf{X}_+ = \mathbf{X}_+ \tag{3}$$

where the unity root matrix $\mathbf{A}$ is defined as follows, in terms of three unity roots P, Q, R and their 'conjugates' $\overline{P}, \overline{Q}, \overline{R}$:

$$\mathbf{A} = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix} \tag{4}$$

The unity roots, also known as primitive roots in number theory [6], are defined by the following congruences:

$$P^n \equiv 1 \ (\text{mod } x),\ Q^n \equiv 1 \ (\text{mod } y),\ R^n \equiv -1 \ (\text{mod } z) \tag{5.5a}$$

$$\overline{P}^n \equiv 1 \ (\text{mod } x),\ \overline{Q}^n \equiv 1 \ (\text{mod } y),\ \overline{R}^n \equiv -1 \ (\text{mod } z) \tag{5.5b}$$

$$P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0),\ \overline{P}, \overline{Q}, \overline{R} \in \mathbb{Z},\ (\overline{P}, \overline{Q}, \overline{R}) \neq (0,0,0) \tag{5.5c}$$

In addition, $\overline{P}, \overline{Q}, \overline{R}$ relate to their standard forms, P, Q, R, by the following 'conjugate relations'

$$\overline{P} \equiv P^{n-1} \ (\text{mod } x),\ \overline{Q} \equiv Q^{n-1} \ (\text{mod } y),\ \overline{R} \equiv -R^{n-1} \ (\text{mod } z) \tag{6}$$

Eigenvector equation (3) mandates one of the three possible eigenvalues (λ_1) is unity. The general form of eigenvector equation for eigenvalue λ is defined as usual by

$$\mathbf{AX}_+ = \lambda \mathbf{X}_+ \tag{7}$$

By defining a ‘Kinetic’ term K and ‘Potential’ term V in terms of the unity roots as

$$K = P\overline{P} + Q\overline{Q} + R\overline{R} \tag{8a}$$

$$V = PQR + \overline{P}\overline{Q}\overline{R} \tag{8b}$$

then the characteristic equation for $\mathbf{A}$ (4) is given by

$$\lambda^3 - K\lambda - V = 0 \tag{9}$$

For the specific, unity eigenvalue $\lambda_1 = 1$ this reduces to

$$1 = K + V,\ \lambda_1 = 1 \tag{10}$$

which is treated as an energy conservation equation (unity total energy) in [1], hence the use of the terms Potential and Kinetic.

The other two eigenvalues values, λ_2 and λ_3, required to complete the set of three, and soon to be identified as eigenvalues of the third component of Isospin ($\mathbf{I}_z$) and Hypercharge ($\mathbf{Y}$) are given by the quadratic solution

$$\lambda_{2,3} = \frac{-1 \pm \sqrt{1 - 4V}}{2} \tag{11}$$

If the Potential V is written in the following form, for arbitrary integer S,

$$V = -S(S+1),\ S \in \mathbb{Z} \tag{12}$$

then the discriminant $\sqrt{1-4V}$ in (11) is a perfect square, and the other two eigenvalues are also integer given in terms of S by

$$\lambda_2 = -1-S\ ,\ \lambda_3 = S \tag{13}$$

Note that all three eigenvalues sum to zero, i.e. $\lambda_1 + \lambda_2 + \lambda_3 = 1 + (-1-S) + S = 0$, as expected for the zero trace matrix $\mathbf{A}$ (4).

Given the form of V (8b) in terms of the unity roots (5), it is not generally possible to write V as a perfect square, and the eigenvalues λ_2 and λ_3 are generally irrational. Furthermore, if V is greater than zero, they are complex. It is of note that the real part of the complex eigenvalues is $-\frac{1}{2}$ as in $\lambda_{2,3} = -\frac{1}{2} \pm bi$, where $b = \sqrt[+]{4V-1}$, and thus provides an important link to the Riemann Hypothesis [7]. This paper thus connects Fermat's last Theorem, Quarks and the Riemann Hypothesis via solutions to (1).

Returning to the eigenvalues (13), there are two very special cases, namely $S=0$ and $S=1$, that lead directly to the Isospin and Hypercharge eigenvalues respectively of the Up, Down and Strange quarks, i.e.

$$S = 0 \Rightarrow V = 0 \tag{14}$$

Third component of Isospin I_z (to within a factor 1/2)

$\lambda_1 = 1$, Up quark, $I_z = \frac{1}{2}$

$\lambda_2 = -1$, Down quark, $\sim I_z = -\frac{1}{2}$

$\lambda_3 = 0$ Strange quark, $\sim I_z = 0$

Note that the other three quarks Charm, Bottom and Top also have zero Isospin as per the Strange quarks, see also SU(6) [2].

$$S = 1 \Rightarrow V = -2 \tag{15}$$

Hypercharge Y (to within a factor 1/3)

$\lambda_1 = 1$, Up quark (1/3)

$\lambda_2 = 1$, Down quark (1/3)

$\lambda_3 = -2$ Strange quark (-2/3)

Note that the other three quarks Charm, Bottom and Top have integer values of 4,-2,4 respectively and are not covered in this SU(3) scheme.

By setting S to either zero or unity this may make it seem easy to acquire these eigenvalues or, indeed, any values the reader may wish to choose, subject to a zero trace for $\mathbf{A}$ (4). But, to reiterate, given the definition of V (8b), it is far from trivial, if actually possible (no proof offered here), to obtain a set of six unity roots $\{P, Q, R, \overline{P}, \overline{Q}, \overline{R}\}$ (5), that give a desired integer value of V such that the discriminant $\sqrt{1-4V}$ in (11) is a perfect square. Except, and this is why the aforementioned link to Isospin and Hypercharge is not just wishful thinking, the two Potential values of $V = 0$ for I_z and $V = -2$ for Y represent very special solutions to (1) for which the Potential energy is the two smallest, absolute, integer values - note that the intermediate values $V = \pm 1$ give irrational eigenvalues, and therefore invalid in this integer-only theory.

Isospin

If the unity roots satisfy the following 'Pythagoras conditions' (for reasons that will become apparent)

$$\overline{P} = P\,,\ \overline{Q} = Q\,,\ \overline{R} = -R\,,\ \text{the Pythagoras Conditions} \tag{16}$$

then it is seen that $V = 0$ (8b) and the eigenvalues are those of I_z (14). The unity root matrix then becomes

$$\mathbf{A}_0 = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix} \tag{17}$$

where $\mathbf{A}_0 = \mathbf{A}$ (4) under Pythagoras Conditions (16)

If one expands out the eigenvector equation (3) using $\mathbf{A}_0$, then

$$x = Ry + Qz \tag{18a}$$

$$y = -Rx + Pz \tag{18b}$$

$$z = Qx + Py \tag{18c}$$

By multiplying (18a) by x, (18b) by y, and (18c) by z, and summing all three, one arrives at Pythagoras, i.e.

$$0 = x^2 + y^2 - z^2 \tag{19}$$

Comparing this with (1) it is seen that this must be the solution for the quadratic exponent $n = 2$, and constant k is thus zero, i.e. this is the Pythagorean solution to (1). It is under these special Pythagoras conditions (16) that most of the Physics in [1] and [2] arises.

By defining the three new variables α, β and γ by

$$(1 - P\overline{P}) = \alpha x\,,\ (1 - Q\overline{Q}) = \beta y\,,\ (1 - R\overline{R}) = \gamma z\,,\ \alpha, \beta, \gamma \in \mathbb{Z} \tag{19a}$$

then under Pythagoras conditions (16) these become

$$(1-P^2)=\alpha x\,,\ (1-Q^2)=\beta y\,,\ (1+R^2)=\gamma z \tag{19b}$$

and the full eigenvector solution [1], for each unique eigenvalue is now given by

$$\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \lambda_1 = 1 \tag{20a}$$

$$\mathbf{X}_0 = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix},\ \lambda_3 = 0 \tag{20b}$$

$$\mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix},\ \lambda_2 = -1 \tag{20c}$$

Note that the row vector $\mathbf{X}^+$ defined by

$$\mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix} \tag{20d}$$

also satisfies the unity eigenvalue equation $\mathbf{X}^+\mathbf{A}_0 = \mathbf{X}^+$ and, in addition, the inner product $\mathbf{X}^+\mathbf{X}_+$ satisfies the following conservation equation in the Potential energy V (8b):

$$\mathbf{X}^+\mathbf{X}_+ = 2+V \tag{20e}$$

See [1] for more details on several such conservation equations between the eigenvectors.

There is an infinite set of such solutions (20), and some small example numbers are tabulated below

x	y	z	P	−Q	R	α	β	−γ
4	3	5	-1	-2	-2	0	-1	-1
12	5	13	-7	-4	-8	-4	-3	-5
8	15	17	-1	-4	-4	0	-1	-1
24	7	25	-17	-6	-18	-12	-5	-13
20	21	29	-11	-13	-17	-6	-8	-10
40	9	41	-31	-8	-32	-24	-7	-25

It is noted that both triples (x,y,z) and (α,β,γ) in eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ respectively, satisfy Pythagoras (19), whereas the unity root roots (P,Q,R), eigenvector $\mathbf{X}_0$, satisfy a hyperbolic equation $1=P^2+Q^2-R^2$. The eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ are also associated with the Up and Down quarks, respectively [2], in accordance with their respective Isospin eigenvalues $I_z = \frac{1}{2}$ and $I_z = -\frac{1}{2}$ (14), whereas the Strange quark, with zero Isospin $I_z = 0$, is associated with the hyperbolic eigenvector $\mathbf{X}_0$. This is a serious point because this eigenvector representation makes the Strange quark algebraically (Hyperbolic) different to the Up and Down quarks (Pythagorean). In [2] the same statement extends to the other three known quarks, Charm, Bottom and Top, and asserts they too are represented by hyperbolic eigenvectors, not Pythagorean. Whilst the Up and Down quarks form stable particles, the other four do not, and this difference in stability is attributed in [2] to the algebraic forms of the eigenvectors.

Hypercharge

The above associates Isospin eigenvalues I_z (14) to a quadratic form of (1), in effect reducing Fermat's Last Theorem to the Pythagoras equation. Nevertheless, Hypercharge operators and their eigenvalues are also given under the same, quadratic Pythagorean scheme, as too other charges such as strangeness and Charm, see [2]. In fact, the Pythagorean work in [2] constructs the Hypercharge matrix operator equivalent ($\mathbf{Y}$) of the Isospin matrix operator $\mathbf{A}_0$ (17) in a contrived way using the Spectral Theorem [8], and asserting, beforehand, the eigenvalues to be those of Hypercharge (15). Thus, it might seem that higher order exponent ($n \geq 3$) forms of (1) are not required to cover all quark charges. However, this contrivance is not actually necessary, and integer Hypercharge eigenvalues occur as a natural solution to (1) for all odd exponents $n \geq 3$ as follows.

The most trivial unity root values satisfying (5) are when all roots have a unity magnitude, i.e.

$$\overline{P} = P = 1,\ \overline{Q} = Q = 1,\ \overline{R} = R = -1 \tag{21}$$

Whilst these values look like they satisfy the Pythagoras conditions (16), for a unity value, they do not because here $\overline{R} = R$ as opposed to $\overline{R} = -R$ (16). Just this simple twist is enough to dramatically alter the solution.

Firstly, inserting the unity root values (21) into V (8b) gives $V = -2$ for which $S = 1$ (12) and thus the Hypercharge eigenvalues (15). Secondly, inserting (21) into $\mathbf{A}$ (4) gives

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \tag{22}$$

and expanding out the eigenvector equation (3) for the unity eigenvalue $\lambda_1 = 1$, using (22), then gives the same solution for all three equations, i.e.

$$z = x + y \tag{23}$$

The eigenvector solution for $\lambda_1 = 1$, thus has two arbitrary parameters x and y, where z is now the sum of x and y as in:

$$\mathbf{X}_1 = \begin{pmatrix} x \\ y \\ x+y \end{pmatrix}, \lambda_1 = 1,\ z = x + y \tag{24}$$

For obvious reasons, this is known as the 'z equals x plus y' solution in [1]. Most importantly, this solution satisfies the Coordinate Equation (1) for all odd exponents $n \geq 3$, whereby the non-zero value of k parametrically varies with x, y and exponent n albeit, for the cubic exponent, k is always equal to the exponent, i.e. $k = 3$ for $n = 3$ as can be seen by inserting for z from (24) into (1):

$$0 = x^3 + y^3 - (x+y)^3 + kxy(x+y) \Rightarrow k = 3 \tag{25}$$

Because the unity eigenvalue is repeated, the two-parameter eigenvector form (24) can be used to give two linearly independent eigenvectors. For example, if $y = x$ and $y = -x$ then two orthogonal eigenvectors are

$$\mathbf{X}_1 = \begin{pmatrix} x \\ x \\ 2x \end{pmatrix}, \lambda_1 = 1,\ y = x \Rightarrow z = 2x \tag{26a}$$

$$\mathbf{X}_2 = \begin{pmatrix} x \\ -x \\ 0 \end{pmatrix}, \lambda_2 = 1,\ y = -x \Rightarrow z = 0 \tag{26b}$$

Lastly, the third eigenvector, for $\lambda_3 = -2$, is of the following form, re-using x as an arbitrary parameter once again:

$$\mathbf{X}_3 = \begin{pmatrix} x \\ x \\ -x \end{pmatrix}, \lambda_3 = -2 \tag{27}$$

Note that it does not satisfy $z = x + y$ (23) nor the Coordinate Equation (1)

Once again, as mentioned further above in the case of Isospin, the Up and Down quark, both with a repeated, unity Hypercharge eigenvalue $\lambda_{1,2} = 1$, have an eigenvector representation that is algebraically different to that of the Strange quark, i.e. the Up and Down quarks (15) satisfy both $z = x + y$ (23) and the Coordinate Equation (1), whereas the Strange quark does not satisfy either.

Summary and Conclusions

It has been shown that a generalised version of Fermat's Last Theorem, known as the Coordinate Equation (1), has two very special solutions:

1) a Pythagorean solution (19) for the smallest possible, zero Potential energy $V = 0$, where the Coordinate Equation reduces to a quadratic exponent with integer Isospin eigenvalues (14);

2) The $z = x + y$ (23) solution, which gives the integer Hypercharge eigenvalues (15) and is valid for any arbitrary odd exponent $n \geq 3$ in (1). This latter solution is obtained for the next smallest magnitude, integer Potential energy solution $V = -2$. Alternatively stated, the two smallest magnitude Potential energy values gives two of the most general and simple solutions to a modified form of Fermat's Last Theorem.

It has been noted that the Up and Down quarks, forming relatively stable particles, i.e. the nucleons, are represented by eigenvectors that algebraically differentiate themselves from the Strange quark, which forms unstable particles.

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