An Overview of Unity Root Matrix Theory 2017

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Abstract. This paper is an overview of Unity Root Matrix Theory (URMT) as developed from its inception around 2009. It begins with URMT's origins in number theory, and a modified form of Fermat's Last Theorem that has an infinite set of integer solutions given as eigenvectors to a unity root matrix. Such solutions are derived by application of an invariance principle to the characteristic eigenvalue equation for the unity root matrix, and classed as an energy conservation equation. The corresponding, three-dimensional eigenvector solution is related to dynamical quantities and, ultimately, quantum mechanics by virtue of URMT's inherent unitary nature. A five-dimensional eigenvector solution is also presented, representing a four-vector event in Special Relativity, and this solution is shown to possess Hubble-like expansion. A variant of this solution is also linked to the relativistic energy-momentum equation. The paper completes with an illustration of the compactification property of URMT's higher dimensional solution and related holographic concepts.

Keywords:

Number Theory and Physics, Physics in Integers, Fermat's Last Theorem, Riemann Hypothesis, Primitive Roots, Unity Roots, Integers Matrices, Eigenvalues, Eigenvectors, Variational Methods, Pythagorean Triples, Pythagorean n-tuples, Quantum Mechanics, Special Relativity, Hubble Law, Compactification, Holography.

Acronyms and Abbreviations

DCE : Dynamical Conservation Equation
FLT : Fermat's Last Theorem
QM : Quantum Mechanics
QPI : Quantum Physical Interpretation
SPI : Standard Physical Interpretation
STR : Special Theory of Relativity
URMT : Unity Root Matrix Theory
URMn : the $n \times n$ matrix formulation of URMT, $n \geq 2$.

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1 Origins in Number Theory

URMT began with an equation, called the 'coordinate equation', originating from Fermat's Last Theorem (FLT). This coordinate equation is given in terms of three integers variables x, y, z, for some integer k, as

$$0 = x^n + y^n - z^n + xyz.k\,. \tag{1.0}$$

The full list of conditions is

$$x, y, z, k, n \in \mathbb{Z},\ n \geq 2,\ 1 < x < y < z,\ \gcd(x, y) = \gcd(y, z) = \gcd(z, x) = 1 \tag{1.1}$$

Furthermore the exponent *n* is also restricted to prime only herein, to keep things simple. It could be composite, but this is not necessary for illustrative purposes.

Were *k* to be zero, then this equation would be identical to the famous equation of Fermat's Last Theorem (FLT), i.e.

$$0 = x^n + y^n - z^n, \text{FLT},\ k = 0\,. \tag{1.2}$$

Of course, if there are to be integer solutions x, y, z, then k is never zero for $n > 2$, as proven by Wiles [1]. However, unlike FLT, this equation has solutions, in integers, for non-zero *k*. By defining a vector $\mathbf{X}_{3+}$ comprising the integer solution x, y, z

$$\mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{1.3}$$

then this vector is an eigenvector of URMT's founding unity root matrix $\mathbf{A}_3$ (defined next), for unity eigenvalue, i.e.

$$\mathbf{A}_3 \mathbf{X}_{3+} = \mathbf{X}_{3+}\,, \tag{1.4}$$

where the unity root matrix $\mathbf{A}_3$ is defined as follows, in terms of three unity roots P, Q, R and their 'conjugates' $\overline{P}, \overline{Q}, \overline{R}$ (see Appendix (A) for an introduction to unity roots):

$$\mathbf{A}_3 = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix}. \tag{1.5}$$

The unity roots, also known as primitive roots in number theory, are defined by the following congruences:

$$P^n \equiv 1 \pmod{x},\ Q^n \equiv 1 \pmod{y},\ R^n \equiv -1 \pmod{z} \tag{1.6a}$$

$$\overline{P}^n \equiv 1 \pmod{x},\ \overline{Q}^n \equiv 1 \pmod{y},\ \overline{R}^n \equiv -1 \pmod{z}\,. \tag{1.6b}$$

No triplet of unity roots is ever entirely zero, i.e. null.

$$P,Q,R \in \mathbb{Z},\ (P,Q,R) \neq (0,0,0)\,,\ \overline{P},\overline{Q},\overline{R} \in \mathbb{Z},\ (\overline{P},\overline{Q},\overline{R}) \neq (0,0,0)\,. \tag{1.6c}$$

However, contrary to the above definition, it is possible that one or two (but not all three) can be zero in special circumstances, not discussed further herein.

In addition, $\overline{P},\overline{Q},\overline{R}$ relate to their standard forms, P,Q,R, by the following 'conjugate relations'

$$\overline{P} \equiv P^{n-1} \pmod{x}\,,\ \overline{Q} \equiv Q^{n-1} \pmod{y}\,,\ \overline{R} \equiv -R^{n-1} \pmod{z}\,. \tag{1.7}$$

The eigenvector equation (1.4) forms the basis of URMT and, crucially, for a quadratic exponent *n* in (1), leads to a physical interpretation that is the basis of 'Physics in Integers' [2], i.e. the premise that the physical laws are really those of number theory and nothing more. As a taster for this, the URMT eigenvalue *C* (known as 'big *C*') will later be equated with both the speed of light, 'little c', and Planck's constant $\overline{h}$. In other words, physical laws actually arise from the laws of integer arithmetic and that, far from integers approximating the continuous, real-valued (or complex) continuous differential equations of current physical laws, including string theory, it is, in fact, the reverse, with real numbers considered to approximate what is ultimately a discrete, integer-only theory of nature. Taken to its extreme, nature follows the laws of number theory and, in particular, for a quadratic exponent, sums and squares its way into existence.

The eigenvector connection between solutions to (1.0) and unity root matrices appears to be a URMT-unique result, and original, but that is not to say that this is also published somewhere else, albeit it is highly likely to be phrased in a much different way. However, after many years and public postings, no refutation of originality, or accusation of plagiarism, has been received, neither do unity root matrices in the form (1.5) appear to be the subject of any prior published work. It is noted by the author that there are numerous matrices containing the complex roots of unity, e.g. weight matrices used in discrete Fourier transforms, but these are not the same, or isomorphic, to those unity root matrices studied in URMT. Lastly, extensive numerical studies have not revealed any errors or contradictions.

2 Example

This is the classic, cubic example (9,31,70) given in [2]#1.

$$n = 3\,,\ x = 9\,,\ y = 31\,,\ z = 70 \tag{2.0a}$$

The coordinate equation (1.0)

$$0 = 9^3 + 31^3 - 70^3 + 16.9.31.70\,,\ k = 16\,. \tag{2.0b}$$

Dynamical variables P,Q,R (1.6a) and their conjugates $\overline{P},\overline{Q},\overline{R}$ (1.6b)

$$P = -2\,,\ Q = -6\,,\ R = -11\,, \tag{2.1a}$$

$$\overline{P} = +4\,,\ \overline{Q} = +5\,,\ \overline{R} = +19\,. \tag{2.1b}$$

The unity root matrix (1.5)

$$\mathbf{A}_3 = \begin{pmatrix} 0 & -11 & +5 \\ +19 & 0 & -2 \\ -6 & +4 & 0 \end{pmatrix}. \tag{2.2a}$$

The eigenvector solution (1.3)

$$\mathbf{X}_{3+} = \begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix}. \tag{2.2b}$$

The eigenvector equation (1.4), 'dynamical equations', in matrix form

$$\mathbf{A}_3\mathbf{X}_{3+} = \mathbf{X}_{3+}, \begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix} = \begin{pmatrix} 0 & -11 & +5 \\ +19 & 0 & -2 \\ -6 & +4 & 0 \end{pmatrix}\begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix}. \tag{2.2b}$$

Unity Root Properties (1.6)

$$\begin{gathered} P^n \equiv +1 \pmod{x},\ -2^3 \equiv +1 \pmod{9}. \\ Q^n \equiv +1 \pmod{y},\ -6^3 \equiv +1 \pmod{31}. \\ R^n \equiv -1 \pmod{z},\ -11^3 \equiv -1 \pmod{70}. \\ \overline{P}^n \equiv +1 \pmod{x},\ +4^3 \equiv +1 \pmod{9}. \\ \overline{Q}^n \equiv +1 \pmod{y},\ 5^3 \equiv +1 \pmod{31}. \\ \overline{R}^n \equiv -1 \pmod{z},\ +19^3 \equiv -1 \pmod{70}.. \end{gathered} \tag{2.3}$$

Conjugate Relations (1.7)

$$\begin{gathered} \overline{P} \equiv P^{n-1} \pmod{x},\ +4 \equiv (-2)^2 \pmod{9}. \\ \overline{Q} \equiv Q^{n-1} \pmod{y},\ +5 \equiv (-6)^2 \pmod{31}. \\ \overline{R} \equiv -R^{n-1} \pmod{z},\ +19 \equiv -(-11)^2 \pmod{70}. \end{gathered} \tag{2.4}$$

The kinetic term K , see (4.3) further below,

$$K = P\overline{P} + Q\overline{Q} + R\overline{R} = -247. \tag{2.5}$$

The Potential term V, see (4.4) further below,

$$V = PQR + \overline{P}\,\overline{Q}\,\overline{R} = 248. \tag{2.6}$$

The Dynamical Conservation Equation, see (4.6) further below, for unity eigenvalue $C = 1$

$$1 = K + V = -247 + 248 . \tag{2.7}$$

The divisibility factors α , β and γ , see (4.16) and (4.18) further below, for unity eigenvalue $C = 1$

$$\begin{aligned} (1 - P\overline{P}) &= \alpha x \,,\ \alpha = +1 . \\ (1 - Q\overline{Q}) &= \beta y \,,\ \beta = +1 . \\ (1 - R\overline{R}) &= \gamma z \,,\ \gamma = +3 . \end{aligned} \tag{2.8}$$

The co-vector $\mathbf{X}^{3+} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$, see (4.16) further below,

$$\mathbf{X}^{3+} = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}. \tag{2.9}$$

The dual dynamical equations $\mathbf{X}^{3+}\mathbf{A}_3 = \mathbf{X}^{3+}$ in matrix form, see (4.17) further below, for unity eigenvalue $C = 1$

$$\begin{pmatrix} 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & -11 & +5 \\ +19 & 0 & -2 \\ -6 & +4 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 3 \end{pmatrix}. \tag{2.10}$$

The Potential Equation in vector form, see (4.20b) further below, for unity eigenvalue $C = 1$

$$\mathbf{X}^{3+}\mathbf{X}_{3+} = 2 + V \,,\ \begin{pmatrix} 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix} = 250 = 2 + 248 . \tag{2.11}$$

3 A Link to the Riemann Hypothesis.

Whilst the single eigenvalue of unity, as given by (1.4), is key to URMT and its unity roots, the other two eigenvalues are of note to those interested in the Riemann Hypothesis. This is because the unity root matrix $\mathbf{A}_3$ (1.5) has a zero trace, i.e. the leading diagonal sums to zero, in fact it is all-zero. From linear algebra [3], the sum of the eigenvalues is given by the trace, and hence URMT's eigenvalues sum to zero. Since the first eigenvalue is unity ($\lambda_1 = 1$), this means the other two eigenvalues (λ_2, λ_3) must sum to minus one, i.e.

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \,,\ \lambda_1 = 1 \ \Rightarrow \lambda_2 + \lambda_3 = -1 \tag{3.1}$$

If these other two eigenvalues are complex then they must be of the following form for some real-valued *b*, i.e.

$$\lambda_2 = -\frac{1}{2} + bi \,,\ \lambda_3 = -\frac{1}{2} - bi \,,\ b \in \mathbb{R},\ b \neq 0 . \tag{3.2}$$

and thus they lie on the 'critical line' $x=-1/2$ in the complex plane $z=x+iy$. In the Riemann hypothesis it is the trivial zeros that lie on the $-1/2$ line, and the non-trivial zeros lie along the positive, $x=1/2$ line. However, it is very easy in URMT to convert any positive unity root eigenvalue solution to a negative solution, simply by reversing the sign of $\mathbf{A}_3$, i.e. $\mathbf{A}_3 \to -\mathbf{A}_3$, which implies $\lambda_1 \to -1$, hence $-1/2 \to 1/2$, and thus the URMT solution now relates directly to the $x=1/2$ line, and gives a link to the Riemann Hypothesis. Indeed, the subject of Physics and the Riemann Hypothesis is a well-explored field with some solid background, e.g. [4]. For a more detailed analysis of the URMT solution and how to vary the value of *b* in (3.2), see [5].

4 Physical Development

In preparation for some physical aspects, the earlier, pertinent equations are re-written for a non-unity eigenvalue 'big C' as follows:

$$\mathbf{A}_3\mathbf{X}_{3+} = C\mathbf{X}_{3+},\ C\in\mathbb{Z},\ C\geq 1. \tag{4.0}$$

Since *C* will be related to a velocity, which can be the speed of light, it is preferable to use symbol *C* rather than unity, so that, in general.

$$P^n \equiv C^n \pmod{x},\ Q^n \equiv C^n \pmod{y},\ R^n \equiv -C^n \pmod{z},\ n\in\mathbb{Z},\ n\geq 2 \tag{4.1a}$$

$$\overline{P}^n \equiv C^n \pmod{x},\ \overline{Q}^n \equiv C^n \pmod{y},\ \overline{R}^n \equiv -C^n \pmod{z} \tag{4.1b}$$

$$C^{n-2}\overline{P} \equiv P^{n-1} \pmod{x},\ C^{n-2}\overline{Q} \equiv Q^{n-1} \pmod{y},\ C^{n-2}\overline{R} \equiv -R^{n-1} \pmod{z} \tag{4.1c}$$

In fact, URMT only requires the unity definition $C=1$, and cases $C>1$ can be reduced to the unity eigenvalue problem by transformation [2]#6. Nevertheless, it is much more convenient, from a physical viewpoint, to use symbol *C*, unity or otherwise, for comparison of URMT equations with those in Physics.

From here onward, the unity roots will be termed 'dynamical variables as they can consistently be ascribed to a velocity quantity, so too the eigenvalue *C*. It is actually more useful to think of their square forms as energy, per unit mass, i.e. for Newtonian kinetic energy *K* then. $v^2 = K/2m$ for velocity *v*, mass *m*.

4.1 The Dynamical Conservation Equation

The usual method to solve eigenvector equations such as (4.0) is via straightforward algebraic manipulation, starting by expanding the matrix equation into three linear equations. However, this is not the URMT way, and a more physical derivation is given shortly in terms of a single energy equation, which is subjected to variational methods, coupled with an invariance principle. This is therefore conceptually similar to the Lagrangian method and the principle of least action.

To ensure any eigenvector equation has a solution, the following non-singular, determinant condition applies to $\mathbf{A}_3$, for arbitrary eigenvalue λ:

$$\det(\mathbf{A}_3 - \lambda I_3) = 0. \tag{4.2a}$$

This expands to the following characteristic equation:

$$0 = -\lambda^3 + (P\overline{P} + Q\overline{Q} + R\overline{R})\lambda + (PQR + \overline{P}\,\overline{Q}\,\overline{R})\,. \tag{4.2b}$$

Of course, one such eigenvalue C, is already specified (4.0). However, its accompanying eigenvector $\mathbf{X}_{3+}$ has not yet been given in terms of the dynamical variables (unity roots), and will be derived shortly (4.13).

Defining a kinetic energy term K and potential energy term V (per unit mass) as follows

$$K = P\overline{P} + Q\overline{Q} + R\overline{R}\,, \text{ kinetic energy per unit mass.} \tag{4.3}$$

$$V = (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C\,, \text{ potential energy per unit mass.} \tag{4.4}$$

then (4.2b) shortens to

$$0 = -\lambda^3 + K\lambda + VC\,. \tag{4.5}$$

Note that naming the two expressions, (4.3) and (4.4), as energy terms is justified on the basis of the resulting physical consistency of the equations, derived next, throughout URMT's physical development. In particular, the kinetic term is seen to comprise quadratic degree terms that are the product of a velocity (4.42) and its conjugate, akin to energy terms in the Lagrangian of classical mechanics, using generalised velocities (or momentum) [6]. The form of the Potential is certainly less recognisable but, again, does lead to a physically consistent interpretation as a scalar field with a gradient related to a force, see Section (4.4).

Substituting for the single eigenvalue $\lambda = C$ and dividing throughout by C, which is always positive, non-zero (4.0), then the following 'Dynamical Conservation Equation' is obtained

The Dynamical Conservation Equation (DCE)

$$C^2 = K + V \tag{4.6}$$

$$C^2 = P\overline{P} + Q\overline{Q} + R\overline{R} + (PQR + \overline{P}\,\overline{Q}\,\overline{R})/C$$

By virtue of its name, this is a true conservation equation and one of several in URMT. Consequently, the eigenvalue *C* is also known as the 'invariant eigenvalue'.

This DCE is the founding equation for the physical development of the theory. Indeed, barring the explicit omission of mass, it can already be seen where URMT is headed with (4.6), i.e. multiplying by the mass m gives

$$E = mC^2 = m(K + V)\,,\ K \text{ and } V \text{ are per unit mass,} \tag{4.7}$$

$$E = mc^2\,,\ C = c\,. \tag{4.7b}$$

This energy equation pervades all URMT, as will be seen throughout this paper, and offers some justification for ascribing the terms (4.3) and (4.4) to kinetic and potential energies per unit mass, i.e. velocity squared. The interpretation of the dynamical variables (unity roots) as velocity quantities is justified by the consistent physical results it gives, and is known as the Standard Physical Interpretation (SPI) detailed again in Section (4.9). There is also a related, more recent Quantum Physical Interpretation (QPI) detailed in Section (4.13).

In general, under the SPI, invariant eigenvalue *C* is equated to a velocity quantity, and it is easily seen that if it were the speed of light, (4.7) would be that of Einstein's famous equation. This aspect takes on a more solid basis when the three-dimensional, 3x3 matrix theory, that is specifically known as URM3, becomes a 5x5 matrix theory known as URM5, which is the basis of URMT's relativistic connections, Section (5). There is also a 2x2 theory, URM2, which is the basis of quantum-mechanical 'spin' in URMT [7] (think 2x2 Pauli spin matrices [8]). In relation to spin, URM3 itself has another physical re-interpretation where the dynamical variables are angular velocities instead of linear velocities, and then it becomes a theory of angular momentum. This latter interpretation was actually the basis for naming the unity roots P,Q,R as dynamical variables, since it was seen that they are indeed related to the commonly-used (in aerospace at least), body rates p,q,r.

4.2 Variational Methods

The DCE (4.6) is actually solved for its eigenvectors by a variational method, unique to URMT, which gives both the equations and their solution. Contrast this with the Lagrangian approach (variational calculus), which generally gives just the equations of motion that are then solved separately.

Local variations

Because the dynamical variables are unity roots, defined by congruences (4.1), they are not unique, and their congruence definitions hold true for three, arbitrary, integer 'local' variations η,δ,ε

$$\begin{gathered} P \to P + \delta x\,,\ \overline{P} \to \overline{P} - \varepsilon x\,, \\ Q \to Q + \varepsilon y\,,\ \overline{Q} \to \overline{Q} - \eta y\,, \\ R \to R + \eta z\,,\ \overline{R} \to \overline{R} - \delta z\,, \\ \eta,\delta,\varepsilon \in \mathbb{Z}. \end{gathered} \tag{4.8}$$

Note that there is also a 'global Pythagoras' variation, detailed later in Section (4.6).

Rewinding briefly to the original coordinate equation (1.0), and its solution *x*,*y*,*z*, it is seen that the dynamical variables (unity roots), are not present and, for every solution, there are therefore an infinite set of dynamical variables as parameterised by the above three local variations η,δ,ε. Thus, when applying these local transformations (4.8) to the DCE (4.6), the DCE itself must remain invariant to these variations with a conserved value of C^2 and, upon its expansion, all terms in η,δ,ε must therefore equate to zero. Doing this gives six separate equations, one for each of the six possible combinations of the local variational elements η,δ,ε. Collecting the three quadratic variations in $\delta\varepsilon$, $\eta\varepsilon$ and $\eta\delta$, gives the following three linear equations:

$$\begin{aligned} Cx &= Ry + \overline{Q}z\,,\ \delta\varepsilon \text{ term}, \\ Cy &= \overline{R}x + Pz\,,\ \eta\varepsilon \text{ term} \\ Cz &= Qx + \overline{P}y\,,\ \eta\delta \text{ terms} \end{aligned} \tag{4.9}$$

These three equations are nothing more than the eigenvector matrix equation (1.4) written out in full to give three linear equations. Thus, this variational method has provided the Lagrangian-equivalent of the equations of motion and, consequently, the three linear equations (4.9) are known hereafter in URMT as the dynamical equations.

As for the actual solution to these dynamical equations, the remaining three, linear (first degree) variational terms in $\eta, \delta, \varepsilon$ give the three possible solutions in x, y, z to these equations, two of which are independent:

$$z(C\overline{R} + PQ) = y(CQ + \overline{R}\overline{P}),\ \eta \text{ term},\ z \text{ in terms of } y, \tag{4.10a}$$

$$z(CR + \overline{P}\overline{Q}) = x(C\overline{P} + QR),\ \delta \text{ term},\ z \text{ in terms of } x, \tag{4.10b}$$

$$y(C\overline{Q} + RP) = x(CP + \overline{Q}\overline{R}),\ \varepsilon \text{ term},\ y \text{ in terms of } x. \tag{4.10c}$$

One possible eigenvector solution for $\mathbf{X}_{3+}$ can be obtained from (4.10b) and (4.10c) in terms of x as a parameter:

$$\mathbf{X}_{3+} = x.\begin{pmatrix} 1 \\ \left(\dfrac{CP + \overline{Q}\overline{R}}{C\overline{Q} + RP}\right) \\ \left(\dfrac{C\overline{P} + QR}{CR + \overline{P}\overline{Q}}\right) \end{pmatrix}. \tag{4.13}$$

See [2]#5, Appendix (C) for a full list of the nine possible forms of eigenvector solution (three of which are linearly independent), given for a unity eigenvalue $C = 1$.

Whilst the equations above have been written for eigenvalue *C*, ,i.e. $\lambda = C$, they are valid for all three eigenvalues simply by replacing *C* with the other two eigenvalues. These have not been given yet, excepting the reference to Riemann (3.2), which only concerns the other two eigenvalues.

The analytic form (4.13) of the eigenvector $\mathbf{X}_{3+}$, eigenvalue $\lambda = C$, is not actually required further, and its general form $\mathbf{X}_{3+}$ (1.3) is sufficient to proceed.

4.3 The Invariance Principle

Like all good physical theories, URMT does have a founding principle, which is basically a restatement of the above mathematical variational technique, namely that all equations in the dynamical variables, plus the eigenvector $\mathbf{X}_{3+}$ and eigenvalue C, remain invariant to these variations, and this is enshrined in URMT in the following invariance principle:

The Invariance Principle (4.15)

The dynamical equations and their solutions are invariant to a coordinate transformation in the dynamical variables.

where the dynamical equations are those given by (4.9), and the coordinate transformation in the dynamical variables is given by the above local variations (4.8).

In fact, one can actually develop URMT from the DCE (4.6) acting as a founding Lagrangian (technically a Hamiltonian since it is written as the sum of kinetic and potential energies), by applying the Invariance Principle and ultimately deriving the founding coordinate equation (1.0). This process is very similar to mathematical physics, whereby the physical laws came historically first, and then

variational calculus enabled them to be derived from a founding principle, i.e. The Principle of Least Action [6]. This latter method was then used in quantum field theory whereby the Lagrangian itself usually comes first, primarily constructed by adherence to physical symmetry arguments, and from which the field equations of the underlying physical theory are then determined and solved for hopefully meaningful (physically realistic) equations [9].

Thus, to summarise so far, URMT starts with the integer coordinate equation (1.0), which has a solution given by the eigenvector equation (4.0), and this solution is a function of the dynamical variables (4.1). The non-singular condition on this eigenvector equation gives the Dynamical Conservation Equation (4.6), which acts as a Lagrangian such that, when a variational method is applied (4.8), in accordance with the Invariance Principle (4.15), the dynamical equations (4.9) and their eigenvector solution (4.10) can be obtained.

4.4 The Potential Equation

The column eigenvector $\mathbf{X}_{3+}$ (1.3) has a dual, row eigenvector form (upper indices for row-vectors, lower for column vectors) which is defined in terms of three, integer, 'scale factors' α, β, γ as

$$\mathbf{X}^{3+} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}, \qquad (4.16)$$
$$\alpha, \beta, \gamma \in \mathbb{Z},\ (\alpha, \beta, \gamma) \neq (0,0,0).$$

and satisfies the following eigenvector equation, also for eigenvalue C,

$$\mathbf{X}^{3+}\mathbf{A}_3 = C\mathbf{X}^{3+} \qquad (4.17)$$

Most importantly, the scale factors α, β, γ relate to the coordinates x, y, z by the 'divisibility relations'

$$\begin{aligned} (C^2 - P\overline{P}) &= \alpha x \\ (C^2 - Q\overline{Q}) &= \beta y \\ (C^2 - R\overline{R}) &= \gamma z \end{aligned} \qquad (4.18)$$

Summing all three divisibility relations gives

$$3C^2 - (P\overline{P} + Q\overline{Q} + R\overline{R}) = \alpha x + \beta y + \gamma z \qquad (4.19)$$

and, by substituting for the kinetic energy K (4.3) and using the DCE (4.6), this becomes the 'potential equation':

$$2C^2 = \alpha x + \beta y + \gamma z - V \text{, the potential equation.} \qquad (4.20)$$

Using the definitions of $\mathbf{X}_{3+}$ (1.3) and $\mathbf{X}^{3+}$ (4.16), this is written in eigenvector, inner product form as

$$\mathbf{X}^{3+}\mathbf{X}_{3+} = \alpha x + \beta y + \gamma z = 2C^2 + V. \qquad (4.20b)$$

This is another conservation equation in URMT, and links all four sets of variables x, y, z, P, Q, R, $\overline{P}, \overline{Q}, \overline{R}$ and α, β, γ to give another invariant $2C^2$ to go with C^2 (the total energy per unit mass) in the DCE (4.6).

Some physical justification for calling (4.20) a potential (energy) equation and, indeed, assigning the name potential in (4.4), can be seen by noting that the scale factors α, β, γ are later physically associated with position coordinates, and the coordinates x, y, z as accelerations - or rather negative force per unit mass; see Section (4.9). With this in mind, by defining a gradient operator as

$$\nabla = (\partial_\alpha, \partial_\beta, \partial_\gamma), \tag{4.21}$$

then applying this to the potential equation (4.20), for constant C (invariant by definition), gives

$$\nabla V = (x, y, z), \tag{4.22}$$

and since x, y, z are associated with accelerations (negative), i.e. force per unit mass ($\mathbf{F}$), then ∇V is consistent with the standard force/potential equation

$$\mathbf{F} = -\nabla V = -(x, y, z), \text{ force per unit mass.} \tag{4.23}$$

Although the form of gradient operator (4.21) is not actually used further, the consistent derivation here of a force from a scalar potential provides some more substance to the rather abstract energy terms and physical associations mentioned above and used throughout.

The scale factors α, β, γ are actually the 'duals' of the coordinates x, y, z, with the former forming the eigenvector $\mathbf{X}^{3+}$ (4.16), and the latter, the eigenvector $\mathbf{X}_{3+}$ (1.3), with their inner product (4.20b) forming an invariant, for a zero potential energy, as will be seen later (4.35). Being the duals of each other means that all equations in URMT are symmetric upon interchange of x, y, z and α, β, γ, see (4.18) for example. However, this does not mean the physical values are identical. Indeed, once a value for the coordinates (accelerations ~ force per unit mass) is specified, then the scale factors (positions) take on a completely different set of values, and any symmetry between the two is broken. An important consequence of this is that there is not an equivalent of the coordinate equation (1.0), for α, β, γ, i.e. if x, y, z satisfy (1.0), which they do by definition, then α, β, γ do not. Nevertheless, one can reformulate URMT by starting with α, β, γ in place of x, y, z in what is known as a dual 'frequency domain' formulation. Although not yet detailed, the standard physical interpretation (SPI) of URMT, Section (4.9), as given in terms of x, y, z, is known as a 'time-domain' formulation, whilst the dual formulation is a 'frequency-domain' formulation. The time-domain aspects are given in Section (4.12), whilst the reader is referred to [7] for the full, time/frequency domain duality.

4.5 A Global Pythagoras Variation

Returning to the variational methods in Section (4.2), by setting the two, local variational parameters η and ε to the value of δ in the following way

$$\eta = \delta, \ \varepsilon = -\delta, \tag{4.24}$$

then the dynamical variables (4.8) now transform as follows:

$$P \to P + \delta x\,,\ Q \to Q - \delta y\,,\ R \to R + \delta z\,, \tag{4.25}$$
$$\overline{P} \to \overline{P} - \delta x\,,\ \overline{Q} \to \overline{Q} - \delta y\,,\ \overline{R} \to \overline{R} - \delta z\,,$$

and substituting for these transformed dynamical variables into the DCE (4.6), collecting terms in δ and δ^2, and equating to zero in accordance with the Invariance Principle, the following two expressions are obtained:

$$\delta \text{ term: } \begin{aligned} 0 = {} & x\left(\frac{QR + \overline{Q}\overline{R}}{C}\right) - y\left(\frac{RP + \overline{R}\overline{P}}{C}\right) + z\left(\frac{PQ - \overline{P}\overline{Q}}{C}\right). \\ & + x(\overline{P} + P) - y(\overline{Q} + Q) + z(\overline{R} - R) \end{aligned} \tag{4.26}$$

$$\delta^2 \text{ term: } 0 = x^2 + y^2 - z^2 - xy\left(\frac{R + \overline{R}}{C}\right) + xz\left(\frac{Q - \overline{Q}}{C}\right) + yz\left(\frac{\overline{P} - P}{C}\right). \tag{4.27}$$

The second, δ^2 term is noteworthy in that nowhere, so far, has a quadratic exponent been asserted, and this variation holds for all exponents $n \geq 2$, not just $n = 2$, yet from this nth order derivation the Pythagoras equation naturally emerges. One might hope that this is a big hint to the nature of FLT!

4.6 Pythagoras Conditions

It can be seen from the δ^2 term (4.27) that if the conjugate dynamical variables $\overline{P}, \overline{Q}, \overline{R}$ are equated to their standard forms P, Q, R as follows:

Pythagoras conditions

$$\overline{P} = P\,,\ \overline{Q} = Q\,,\ \overline{R} = -R\,, \tag{4.28}$$

then (4.27) reduces to the Pythagoras equation, i.e.

$$0 = x^2 + y^2 - z^2\,. \tag{4.29}$$

This means that the elements x, y, z of eigenvector $\mathbf{X}_{3+}$ also satisfy the Pythagoras equation. Whilst a quadratic, Pythagorean exponent $n = 2$ has not been asserted, by enforcing the Pythagoras conditions (4.28) on the dynamical variables, if the coordinates satisfy Pythagoras (4.29) then they cannot simultaneously satisfy a higher order, $n > 2$, form of the coordinate equation (1.0), and the k-value in (1.0) must therefore be zero in this case.

Although not demonstrated here (see (4.44)), the scale factors α, β, γ also satisfy Pythagoras, i.e.

$$0 = \alpha^2 + \beta^2 - \gamma^2\,. \tag{4.30}$$

It was noted earlier above that, whilst α, β, γ are duals of x, y, z, they do not satisfy the coordinate equation (1.0). However, now we see that both x, y, z and α, β, γ satisfy the same Pythagoras equation. Nevertheless, they do not simultaneously have the same value, i.e. they are not the same Pythagorean triples.

If the above was all there was to URMT then, whilst intriguing, it might remain a curio with not much real physics. However, armed with the Pythagoras conditions (4.28) and equations of quadratic degree, this is where most of the real URMT physics starts.

4.7 An Invariant Zero Potential

Applying the Pythagoras conditions (4.28) to the kinetic term K (4.3) and potential term V (4.4), they become

$$K = P^2 + Q^2 - R^2, \tag{4.31}$$

$$V = 0. \tag{4.32}$$

The Potential energy is thus zero and the DCE (4.6) is simply a constant energy, kinetic term:

$$C^2 = K \text{ , the DCE, energy per unit mass,} \tag{4.33}$$

and implies by (4.31) that the dynamical variables P,Q,R satisfy the following hyperbolic conservation equation, which is, once again, a form of the DCE (4.6).

$$C^2 = P^2 + Q^2 - R^2. \tag{4.34}$$

Note that given the energy is all kinetic, with zero potential energy, the URMT solutions are linked to those of a massless particle travelling at the speed of light, and hence a reason to link the eigenvalue C with the speed of light c. This physical link is revisited in more detail later, Section (6), when introducing mass into URMT.

With $V = 0$, and under Pythagoras conditions (4.28), the potential equation (4.20) reduces to

$$2C^2 = \alpha x + \beta y + \gamma z \text{ , the potential equation.} \tag{4.35}$$

The characteristic equation (4.5) also simplifies to

$$0 = -\lambda^3 + K\lambda,$$

and thus, using K (4.33), there are three, symmetric eigenvalues

$$\lambda = \pm C, 0.$$

By applying the conditions (4.28) to the δ term (4.26), another conservation equation is obtained, termed the 'Pythagoras delta equation':

$$0 = yQ + zR - xP \text{ , the Pythagoras delta equation.} \tag{4.36}$$

In fact, all five conservation equations (4.29), (4.30), (4.34), (4.35) and (4.36) are related to the inner products of the eigenvectors of the matrix $\mathbf{A}_{30}$, (4.37) below, and there is also a sixth conservation equation that completes the set; see Section (4.10).

Under Pythagoras conditions, the matrix $\mathbf{A}_3$ simplifies as follows, and is also relabelled $\mathbf{A}_{30}$, where the extra subscript of zero denotes it is subject to these conditions:

$$\mathbf{A}_{30} = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix}. \tag{4.37}$$

In pursuit of the conservation equations as vector inner products, all three eigenvectors are given next.

4.8 The Pythagorean Eigenvectors

Having already defined all the necessary variables, the Pythagorean eigenvectors are stated here without proof, as follows, the reader is referred to [2]#2 for full details:

$$\mathbf{X}_{3+} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \mathbf{X}_{30} = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix},\ \mathbf{X}_{3-} = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}, \tag{4.38}$$

$$\mathbf{A}_{30}\mathbf{X}_{3+} = C\mathbf{X}_{3+},\ \mathbf{A}\mathbf{X}_{30} = 0,\ \mathbf{A}_{30}\mathbf{X}_{3-} = -C\mathbf{X}_{3-}.$$

The reciprocal, row-eigenvectors $\mathbf{X}^{3+}$, $\mathbf{X}^{30}$ and $\mathbf{X}^{3-}$ are defined by the following eigenvector equations:

$$\mathbf{X}^{3+}\mathbf{A}_{30} = C\mathbf{X}^{3+},\ \mathbf{X}^{30}\mathbf{A}_{30} = 0,\ \mathbf{X}^{3-}\mathbf{A}_{30} = -C\mathbf{X}^{3-},$$

and obtained from their standard counterparts using the following '$\mathbf{T}$ operator' and relations

$$\mathbf{T}_3 = \mathbf{T}^3 = \begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \text{ the URM3 } \mathbf{T} \text{ operator} \tag{4.39}$$

$$\mathbf{X}^{3+} = \left(\mathbf{T}^3\mathbf{X}_{3-}\right)^T,\ \mathbf{X}^{30} = \left(\mathbf{T}^3\mathbf{X}_{30}\right)^T,\ \mathbf{X}^{3-} = \left(\mathbf{T}^3\mathbf{X}_{3+}\right)^T, \tag{4.40}$$

which gives

$$\mathbf{X}^{3+} = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix},\ \mathbf{X}^{30} = \begin{pmatrix} P & -Q & -R \end{pmatrix},\ \mathbf{X}^{3-} = \begin{pmatrix} x & y & -z \end{pmatrix}. \tag{4.41}$$

Of most note here is the $\mathbf{T}$ operator (4.39) that, to within sign convention, is identical in form to the Minkowski metric of Special Relativity [10], albeit for two spatial dimensions and one time-dimension, known in the literature as a '2+1' relativistic formulation. In fact, in URM3, this is a single spatial dimension, e.g. the *x* axis, with the observer (laboratory) measured time *t* and proper time τ [11] occupying the other two time dimensions. Whilst this may seem a URMT limitation (one less spatial dimension), it typifies URMT's 'holographic' treatment of physics whereby URMT adds one extra dimension to the classical treatment – the physical reality is thus one-dimension down from the mathematical formulation, hence holographic. Specifically, with respect to relativity, the full four-dimensional (three-spatial, one temporal) variant is embodied in the 5x5 matrix incarnation of URMT known as 'URM5'. In fact, URMT adds this extra dimension because it also adds in the relativistic proper time τ in addition to the laboratory time *t*. All this is illustrated starting in Section (5).

4.9 Standard Physical Interpretation

Before proceeding further, it is worth formally stating the standard physical interpretation (SPI) of all URMT variables, eigenvectors and matrices as given in the table below.

(4.42)

Quantity	Physical Dimensions	Physical Interpretation
$\mathbf{X}_{3+}$, $\mathbf{X}^{3-}$, x, y, z	LT^{-2}	acceleration or force per unit mass
$\mathbf{A}_3$, $\mathbf{A}_{30}$, $\mathbf{X}_{30}$, $\mathbf{X}^{30}$, P, Q, R, $\overline{P}, \overline{Q}, \overline{R}$, C	LT^{-1}	velocity or momentum per unit mass
$\mathbf{X}^{3+}$, $\mathbf{X}_{3-}$, α, β, γ	L	position
δ, t_3	T	time
K, V, C^2	L^2T^{-2}	velocity squared or total energy E ($E = C^2$) per unit mass

Note that the physical units of all reciprocal vectors, $\mathbf{X}^{3+}$ etc. can be determined from the dimensionless $\mathbf{T}$ operator relations (4.40) using the above units for the standard forms.

This is not actually the only physical interpretation, and a modified 'Quantum Physical Interpretation (QPI) is given later when considering URMT and its relation to quantum mechanics (QM). Whilst two physical interpretations may seem one too many, this is not dissimilar to modern theoretical physics whereby, whilst the phenomenology may be different, the underlying mathematics is the same, e.g. forces and symmetry groups – as in the case of three seemingly disparate forces, all mathematically linked by unitary symmetry, specifically the groups U(1) for electromagnetism, SU(2) for the weak force and SU(3) for the strong force [9]. In both URMT and QM this means the inner products between the eigenvectors remain invariant under a unitary transformation – in URMT [7] this unitary transformation is none other than that expressed in the Invariance Principle (4.15).

4.10 Conservation Equations and Invariants

The six key conservation equations of URMT (the first five have already been given earlier), as obtained from the inner product relations between the eigenvectors and their reciprocals, are given here for URM3. Note that the complete set of conservation equations is only valid under URMT Pythagoras conditions (4.28):

$$\mathbf{X}^{3-}\mathbf{X}_{3+} = x^2 + y^2 - z^2 = 0 \text{ Pythagoras (zero norm),} \tag{4.43}$$

$$\mathbf{X}^{3+}\mathbf{X}_{3-} = \alpha^2 + \beta^2 - \gamma^2 = 0 \text{ Pythagoras (zero norm),} \tag{4.44}$$

$$\mathbf{X}^{30}\mathbf{X}_{30} = P^2 + Q^2 - R^2 = C^2 \text{ the DCE,} \tag{4.45}$$

$$\mathbf{X}^{3+}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{3-} = \alpha x + \beta y + \gamma z = 2C^2\text{, the potential equation for } V = 0\text{, (4.32)} \tag{4.46}$$

$$\mathbf{X}^{30}\mathbf{X}_{3+} = \mathbf{X}^{3-}\mathbf{X}_{30} = xP - yQ - zR = 0\text{, the delta equation,} \tag{4.47}$$

$$\mathbf{X}^{30}\mathbf{X}_{3-} = \mathbf{X}^{3+}\mathbf{X}_{30} = \alpha P - \beta Q + \gamma R = 0\text{, the dual delta equation.} \tag{4.48}$$

Equations (4.43) and (4.44) are simply Pythagoras, and the eigenvectors $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$ are referred to as having a 'zero norm', where the norm $\|\mathbf{X}\|$ of a vector $\mathbf{X}$ is defined in URMT as the inner product of a column vector with its reciprocal row vector (4.40), i.e.

$$\|\mathbf{X}_{3+}\|^2 = \mathbf{X}^{3-}\mathbf{X}_{3+} = 0 \tag{4.49}$$

$$\|\mathbf{X}_{3-}\|^2 = \mathbf{X}^{3+}\mathbf{X}_{3-} = 0\,.$$

On the other hand, the magnitude of $\mathbf{X}_{3+}$ and $\mathbf{X}_{3-}$ is non-zero ($2C^2$) but constant, as given by (4.46), likewise for $\mathbf{X}_{30}$ (4.45), which has a magnitude therefore of C^2 (4.45).

Notice the complete duality between x, y, z and α, β, γ in these inner products - the two sets of variables are completely interchangeable, whilst keeping in mind that they don't generally have the same value, and neither are they physically interpreted as the same quantity, with x, y, z accelerations (force per unit mass), and α, β, γ positions. In fact, under the SPI (4.42), the x, y, z are related to α, β, γ by time, as shall shortly be seen in Section (4.12), when time-domain eigenvector evolution is discussed. Conversely, the α, β, γ are related to the x, y, z by frequency. Beforehand exploring the domain evolution, the parametric solution for the eigenvectors is required, in which a single, evolutionary parameter, namely time, is key.

4.11 The Parametric Solution

The URM3 eigenvector problem, under Pythagoras conditions (4.28), is a completely solved problem, with the coordinates x, y, z given by the standard parameterisation for two integers k and l

$$k, l \in \mathbb{Z},\ (k,l) \neq (0,0),\ \gcd(k,l) = 1, \tag{4.50}$$

$$x = 2kl,\ y = (l^2 - k^2),\ z = (l^2 + k^2)\,. \tag{4.51}$$

The dynamical variables P, Q, R and scale factors α, β, γ are obtained in terms of both integers k and l, and two new integers s and u, which are actually solutions to the following linear Diophantine equation [12]:

$$C = ks - lu,\ s, u \in \mathbb{Z}. \tag{4.52}$$

This equation always has a solution since $\gcd(k,l) = 1$ (4.50) and, once a particular solution s' and u' is obtained by algorithmic means [12], then an infinite family of solutions can be generated, denoted here by integers s and u, and parameterised by another arbitrary integer t_3 (soon to be associated with time) as follows:

$$s = s' + t_3 l,\ u = u' + t_3 k,$$

$$t_3 \in \mathbb{Z},\ s', u' \in \mathbb{Z},\ (s', u', t_3) \neq (0,0,0)\,. \tag{4.53}$$

With a solution for s and u obtained, then the dynamical variables and scale factors are given by

$$P = -(ks + lu),\ Q = (ls - ku),\ R = -(ls + ku), \tag{4.54}$$

$$\alpha = -2su\,,\ \beta = (u^2 - s^2)\,, \gamma = (u^2 + s^2)\,. \tag{4.55}$$

4.12 Eigenvector Time-domain Evolution and Calculus

By substituting the parametric solution for P, Q, R (4.54) and α, β, γ (4.55) into the eigenvectors (4.38), the following eigenvector evolution equations can be obtained, where the initial values at time zero, $t_3 = 0$, are superscripted with a prime, i.e. $\mathbf{X}'_{3+} = \mathbf{X}_{3+}(t_3 = 0)$ etc.

$$\mathbf{X}_{3+} = \mathbf{X}'_{3+}\text{, static, no } t_3 \text{ dependence,} \tag{4.56}$$

$$\mathbf{X}_{30}(t_3) = -t_3\mathbf{X}_{3+} + \mathbf{X}'_{30}\,, \tag{4.57}$$

$$\mathbf{X}_{3-}(t_3) = -t_3^2\mathbf{X}_{3+} + 2t_3\mathbf{X}'_{30} + \mathbf{X}'_{3-}\,, \tag{4.58}$$

It can be seen from these that two of the three eigenvectors, $\mathbf{X}_{30}$ and $\mathbf{X}_{3-}$, evolve with t_3 and, in fact, this parameter t_3 is also identical to the negative of the global variation parameter δ in (4.25), i.e. $t_3 = -\delta$, hence the SPI (4.42) classifies δ as a temporal parameter.

Since $\mathbf{A}_{30}$ (4.37) and $\mathbf{X}_{30}$ (4.38) are both functions of P, Q, R, and $\mathbf{X}_{3+}$ (4.38) is a function of x, y, z then, by defining the 'plus' (or 'raising') matrix $\mathbf{A}_+$ in terms of x, y, z by

$$\mathbf{A}_+ = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ -y & x & 0 \end{pmatrix}, \tag{4.59}$$

the matrix $\mathbf{A}_{30}$ itself evolves with a similar equation to $\mathbf{X}_{30}$ (4.57), i.e.

$$\mathbf{A}_{30} = -t_3\mathbf{A}_{3+} + \mathbf{A}'_{30}\,. \tag{4.60}$$

This equation is effectively a matrix algebra form of the Invariance Principle (4.15), and will be of importance shortly when relating URMT to QM.

Note that the justification for call $\mathbf{A}_{3+}$ a 'raising' operator, as used in QM [8], is because it, $\mathbf{A}_{30}$ and another URMT matrix $\mathbf{A}_{3-}$ (a function of α, β, γ not elaborated herein, see [2]#5), satisfy the exact same relations, including 'commutator' relations', as those in used QM. Likewise, the matrix $\mathbf{A}_{3-}$ is a 'lowering' operator in URMT - see [7] for more details on their properties with regard to QM.

Perhaps, most importantly, it can be seen that by taking the time derivative of the three eigenvectors, (4.56) to (4.58), they can be seen to relate to each other by calculus relations, also consistent with their physical interpretations (4.42) as position ($\mathbf{X}_{3-}$) velocity ($\mathbf{X}_{30}$) and acceleration ($\mathbf{X}_{3+}$), to within a scale factor, i.e.

$$\frac{d\mathbf{X}_{3-}}{dt_3} = 2\mathbf{X}_{30}\text{, derivative of position} \propto \mathbf{X}_{30}\text{, velocity} \tag{4.61}$$

$$\frac{d\mathbf{X}_{30}}{dt_3} = -\mathbf{X}_{3+}\text{, derivative of velocity } \propto \mathbf{X}_{3+} \text{ acceleration,} \tag{4.62}$$

$$\frac{d\mathbf{X}_{3+}}{dt_3} = 0\text{, derivative of acceleration = zero (constant acceleration).} \tag{4.63}$$

Indeed, making the following, exact associations between a general position vector **r**, velocity vector **v**, and acceleration vector **a**:

$$\mathbf{a} = -\mathbf{X}_{3+}\text{, constant acceleration, } \mathbf{a} = \frac{dv}{dt} \tag{4.64}$$

$$\mathbf{v} = \mathbf{X}_{30}\text{, velocity, } \mathbf{v} = \frac{dr}{dt} \tag{4.65}$$

$$\mathbf{r} = \frac{\mathbf{X}_{3-}}{2}\text{, position.} \tag{4.66}$$

then the evolution equation (4.58) for $\mathbf{X}_{3-}$ becomes the familiar Newtonian equation for the position of a particle under constant acceleration, i.e.

$$\mathbf{r} = -\frac{t^2}{2}\mathbf{X}_{3+} + t\mathbf{X}'_{30} + \mathbf{X}'_{3-} = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}'t + \mathbf{r}', \tag{4.67}$$

where the superscript prime represents the initial value.

It should be noted that URMT is a discrete theory and so, technically, one should use discrete differences in t_3, and infinitesimals, such as dt_3 in the above derivatives, are really finite differences Δt_3 with a minimum non-zero difference of unity. However, time t_3 is considered to have a smallest unit of at least the Planck time ($\sim 10^{-44}$ s) and for any finite time, even of the order 10^{-20}, the standard calculus derivative d/dt_3 is a good, large t_3 approximation for such discrete differences, and the derivative relations above are thus very close to their continuous, real-valued equivalents.

Of course, this is all, so far, very classical, i.e. Newtonian, but higher-dimensional extensions extend to Special Relativity and, equally importantly, the evolution equations, as re-written in an exponential form (next), highlights a clear link to QM.

Exponentiated Evolution

An eigenvector matrix $[\mathbf{X}]$ is defined as a row vector of the three, column eigenvectors $\mathbf{X}_{3+}, \mathbf{X}_{30}, \mathbf{X}_{3-}$, i.e.

$$[\mathbf{X}] = \begin{pmatrix} \mathbf{X}_{3+} & \mathbf{X}_{30} & \mathbf{X}_{3-} \end{pmatrix}, \tag{4.68}$$

with the initial value at $t = 0$ ($t \sim t_3$) denoted by a superscript prime, as in

$$[\mathbf{X}]' = \begin{pmatrix} \mathbf{X}'_{3+} & \mathbf{X}'_{30} & \mathbf{X}'_{3-} \end{pmatrix},\ t = 0,\ t \sim t_3 \tag{4.69}$$

then an exponentiated evolution matrix $\mathbf{E}_t$ is defined in terms of URMT's plus matrix $\mathbf{A}_{3+}$ (4.59), for time *t*, eigenvalue *C*, by

$$\mathbf{E}_t = \exp\left(t \frac{\mathbf{A}_{3+}}{C} \right), \tag{4.70}$$

where $\mathbf{A}_{3+}$ is invariant (time-independent) by the definition of its constituents x, y, z and eigenvector $\mathbf{X}_{3+}$ (4.56). With $\mathbf{I}$ denoting the 3x3 identity matrix, then this expression expands as per a standard exponential series in terms of multiplicative powers of the matrix $\mathbf{A}_{3+}$, i.e.

$$\exp\left(t \frac{\mathbf{A}_{3+}}{C} \right) = \lim_{n\to\infty} \sum^{n} \mathbf{I} + \left(\frac{t\mathbf{A}_{3+}}{C} \right) + \frac{1}{2!}\left(\frac{t\mathbf{A}_{3+}}{C} \right)^2 + \ldots \frac{1}{n!}\left(\frac{t\mathbf{A}_{3+}}{C} \right)^n, \tag{4.71}$$

Using this evolution matrix $\mathbf{E}_t$, then the initial eigenvector matrix $[\mathbf{X}]'$ evolves according to the matrix product

$$[\mathbf{X}]_t = \mathbf{E}_t [\mathbf{X}]'. \tag{4.72}$$

A key feature in URMT is that the series expansion (4.71) is finite and, in fact, terminates at the second order term because $\mathbf{A}_{3+}$ has the following property that it is zero for cubic and higher order powers, i.e.

$$\mathbf{A}_{3+}^n = 0, \; n \geq 3, \tag{4.73}$$

and so the evolution matrix $\mathbf{E}_t$ series expansion thus reduces to just

$$\mathbf{E}_t = \mathbf{I} + \left(\frac{t\mathbf{A}_{3+}}{C} \right) + \frac{1}{2}\left(\frac{t\mathbf{A}_{3+}}{C} \right)^2. \tag{4.74}$$

Time-domain Evolution and the Wave Function

With URMT evolution now written in an exponential, unitary form, it is intriguing to look at a very similar equation in QM.

For a time-independent Hamiltonian **H**, the wavefunction at time *t*, denoted by $\psi(t)$, is given in terms of its initial wavefunction, time $t = 0$, according to

$$\psi(t) = \exp(-it \frac{\mathbf{H}}{\bar{h}}) \psi(0) \tag{4.75}$$

Given the wavefunction is a state vector, then by comparing this with URMT's own time-domain eigenvector evolution (4.72) the following associations are made between QM and URMT:

$$\psi(t) \sim [\mathbf{X}]_t, \; \psi(0) \sim [\mathbf{X}]', \; -i \frac{\mathbf{H}}{\bar{h}} \sim \frac{\mathbf{A}'_{3+}}{C}. \tag{4.76}$$

The Hamiltonian **H** is time-independent, by definition, and so too the URMT raising operator $\mathbf{A}'_{3+}$. The scaled Planck constant $\bar{h}$ is, of course, also constant by definition, and so too the URMT invariant eigenvalue *C*. From the last term above, the following associations are thus made

$$-i\mathbf{H} = \mathbf{A}'_{3+} \tag{4.76b}$$

$$\bar{h} = C \tag{4.76c}$$

Thus, eigenvalue C can now be directly equated to Planck's constant. Indeed, this is independently verified in [7] where the association of $\bar{h}$ with C is made in connection with quantised angular momentum and spin.

This comparison is quite remarkable given that URMT's time-domain evolution is really a consequence of the Invariance Principle (4.15), which is pure number theory and just an esoteric statement that, due to the congruence relations (4.1), there is an infinite set of solutions parameterised by a numeric parameter, ascribed to time here for time-domain evolution.

4.13 Quantum Physical Interpretation

The SPI can be considered a Newtonian-like physical interpretation in so far as it traditionally talks in terms of kinematic quantities such as acceleration and velocity. However, the above comparison of URMT's time-domain evolution gives an alternative Quantum Physical Interpretation (QPI) where the units of C are now those of Planck's constant, and the units of $\mathbf{A}_{3+}$ are those (per unit mass) of the Hamiltonian (energy) i.e.

$$units(\mathbf{A}_{+}) = units(\mathbf{H}) = J = L^2T^{-2}\text{, QPI energy, per unit mass} \tag{4.77a}$$

$$units(C) = units(\bar{h}) = JT = L^2T^{-1}\text{, QPI action (see below), per unit mass} \tag{4.77b}$$

with the ratios $\mathbf{A}_{3+}/C$ and $\mathbf{H}/\bar{h}$ possessing the units of frequency.

This newer QPI is really just a slight adjustment to the more established SPI, Section (4.9), which primarily speaks in terms energy-related quantities rather than action quantities, as discussed shortly. The 'slight adjustment' is actually a multiplication of the equivalent SPI physical dimensions by length 'L', and, because the comparison of $\mathbf{A}_{3+}/C$ with $\mathbf{H}/\bar{h}$ is a ratio, the multiplication of both the numerator and denominator by the same physical dimension of length cancels so that both the SPI and QPI give the same units for these ratios, i.e. frequency or reciprocal time.

The QPI is actually put on a much firmer basis in [7] with the URMT representation of Quark Flavour, i.e. how to mathematically represent the six quark flavour states (up, down, strange, charm, bottom and top) as eigenvectors in URMT. In particular, the matrices of URMT (there are actually nine unique matrices as opposed to the two matrices $\mathbf{A}_{30}$ and $\mathbf{A}_{3+}$ detailed so far) are related to the symmetry groups SU(N) [9] via their mathematical commutation relations [3].

A strong reason to use this new QPI is that the units of Planck's constant are those of the dynamical quantity known as action, and this leads to a subsequent reinterpretation of URMT's Invariance Principle in terms of action quantities. The reasoning behind this is as follows:

Given the units of C are those of an action quantity, then so too are those of $\mathbf{A}_{30}$ since C and $\mathbf{A}_{30}$ both have the same units as can be deduced from $\mathbf{A}_{30}\mathbf{X}_{3+} = C\mathbf{X}_{3+}$ (4.38). In addition, since the units of $\mathbf{A}_{3+}$ are now those of energy (4.77), then the quantity $t\mathbf{A}_{+}$ is also that of an action quantity, and so the algebraic form of the Invariance Principle, i.e. $\mathbf{A}_{30} = -t_3\mathbf{A}_{3+} + \mathbf{A}'_{30}$ (4.60), is thus a statement on an action quantity, albeit the principle is not per se a 'principle of least action', but more just a statement on the behaviour of the dynamical, eigenvector equations $\mathbf{A}_{30}\mathbf{X}_{3+} = C\mathbf{X}_{3+}$ (4.38) under a

change '$-t\mathbf{A}'_{3+}$' in action $\mathbf{A}_{30}$ as per (4.60). Looking at the equivalent transformations in the dynamical variables P,Q,R (4.25) for $\delta = -t_3$, the principle states that the $\mathbf{X}_{3+}$ eigenvector solution (4.13) (under Pythagoras conditions (4.28)), comprising coordinates *x*,*y*,*z* (4.38) is invariant to a time-domain change in the action of the dynamical variables.

Most importantly, although not shown, the inner product relations between the URMT eigenvectors, Section (4.10), remain invariant to time-domain evolution, just like unitary transformations on the eigenvector wave functions in QM preserve their inner products, and thus URMT transformations are effectively unitary, but do not explicitly possess the complex-valued nature of QM and its associated 'Hilbert Space' of wavefunctions.

Lastly, although not shown, see [7], with the three quark particles (up, down and strange) represented by URMT eigenvectors ($\mathbf{X}_{3+}, \mathbf{X}_{3-}$ and $\mathbf{X}_{30}$ respectively) the corresponding anti-particles (anti-up, anti-down, anti-strange) are represented by the reciprocal vectors ($\mathbf{X}^{3-}, \mathbf{X}^{3+}$ and $\mathbf{X}^{30}$ respectively), which evolve in a time-reversed manner as follows. By denoting the 'conjugate' matrix $[\overline{\mathbf{X}}]$ as a column vector of the URMT reciprocal row eigenvectors, i.e.

$$[\overline{\mathbf{X}}] = \begin{pmatrix} \mathbf{X}^{3-} \\ \mathbf{X}^{30} \\ \mathbf{X}^{3+} \end{pmatrix}, \tag{4.78}$$

and defining the conjugate (time-reversed) eigenvector evolution matrix $\overline{\mathbf{E}}_t$ as the inverse of $\mathbf{E}_t$ (4.70), i.e.

$$\overline{\mathbf{E}}_t = \mathbf{E}_t^{-1} = \exp\left(-t\frac{\mathbf{A}_{3+}}{C}\right), \tag{4.79}$$

then the reciprocal eigenvectors (rows of $[\overline{\mathbf{X}}]$) evolve according to

$$[\overline{\mathbf{X}}]_t = [\overline{\mathbf{X}}]'\overline{\mathbf{E}}_t . \tag{4.79}$$

By evaluating the series (4.74) for negative time, and expanding (4.78) in terms of the initial reciprocal eigenvectors (4.69), then the reciprocal eigenvectors evolve according to

$$\begin{aligned} \mathbf{X}^{3-} &= \mathbf{X}'^{3-}, \text{ static, no } t_3 \text{ dependence,} \\ \mathbf{X}^{30}(t_3) &= -t_3\mathbf{X}^{3-} + \mathbf{X}'^{30}, \\ \mathbf{X}^{3+}(t_3) &= -t_3^2\mathbf{X}^{3-} + 2t_3\mathbf{X}'^{30} + \mathbf{X}'^{3+}, \end{aligned} \tag{4.80}$$

These same equations can actually be obtained using the standard **T** operator relations (4.40), i.e.

$$[\overline{\mathbf{X}}] = \left(\mathbf{T}[\mathbf{X}]\right)^T . \tag{4.81}$$

Notice now that it is $\mathbf{X}^{3-}$ that is invariant and $\mathbf{X}^{3+}$ evolves with time.

Lastly, whilst the above alluded to a three-quark (up, down and strange) quark representation, URMT actually provides a full, six-quark representation (adding the charm, bottom and top quarks) in its 6x6 incarnation 'URM6' [7].

4.14 Geometric and Physical Aspects

So far, all URMT's properties have been algebraically expressed, but the eigenvector solution also possesses some interesting geometric properties, as now described.

The two eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$, for the two, non-zero eigenvalues $\pm C$, are Pythagorean, i.e. they satisfy the Pythagoras equation (4.29) and (4.30), and have zero norm (4.49). Because they satisfy Pythagoras, they each form a 2D, discrete cone in 3D, ultimately parameterised by three integers k, l (4.50) and t_3 (4.53), where the third parameter t_3 is temporal. The set of all points covered by these parameters represents an infinite set of eigenvectors, and is denoted by the two cone sets $\mathbf{C}_U$ and $\mathbf{C}_L$ for $\mathbf{X}_+$ and $\mathbf{X}_-$ respectively. For each point in $\mathbf{C}_U$, i.e. fixed $\mathbf{X}_+$, the position eigenvector $\mathbf{X}_-$ evolves with time t_3. For large t_3, see (4.58), it changes by multiples of $\mathbf{X}_+$ and, given both it and $\mathbf{X}_+$ are Pythagorean, with a zero norm (4.49), it effectively traces a null trajectory in the cone $\mathbf{C}_L$. Furthermore, this trajectory has inverse square law curvature with respect to time t_3 [2]#3, Since it is also at a zero, constant potential at every point (4.32), there are no forces acting in the direction of motion and it therefore possesses a constant kinetic energy (per unit mass) of C^2. It is thus physically interpreted as the null (zero norm), geodesic trajectory of a massless particle (with C equated to the speed of light *c*) in free-fall.

Because $\mathbf{X}_+$ and $\mathbf{X}_-$ can never be zero, due to the non-zero value of eigenvalue C (4.0), the cones $\mathbf{C}_U$ and $\mathbf{C}_L$ actually have no tip, i.e. there is no point (0,0,0), and this is termed 'no-singularity' in URMT for obvious reasons.

As regards the zero eigenvector $\mathbf{X}_0$, it represents a velocity (4.42), and its solution space forms a 2D, discrete hyperbolic sheet in 3D, denoted by the infinite set of points $\mathbf{H}$ [2]#3. The discrete hyperbolic sheet is the DCE, i.e. the conservation equation (4.45) in $\mathbf{X}_0$, where the elements of $\mathbf{X}_0$ are the dynamical variables P,Q,R. Like $\mathbf{X}_-$, $\mathbf{X}_0$ also evolves with time t_3.

Taken together, the union of the sets $\mathbf{C}_L$, $\mathbf{C}_U$ and $\mathbf{H}$ forms the discrete lattice $\mathbf{L}$, which represents the complete URM3 eigenvector solution. As time t_3 progresses, the discrete hyperbolic sheet of $\mathbf{X}_0$ converges (asymptotically) on to the cone $\mathbf{C}_L$, both of which align anti-parallel to $\mathbf{C}_U$, and the solution is said to 'flatten' [2]#3. At every point in the lattice, the conservation equations, Section (4.10), i.e. inner products between the eigenvectors, give the same set of invariants, $\{0,\pm C^2,\pm 2C^2\}$, and for unity eigenvalue this is just $\{0,\pm 1,\pm 2\}$, i.e. the most basic units possible. Note that the negative values can be achieved by reversing the sign of the $\mathbf{T}$ operator (4.39) without detriment to the eigenvectors (4.41). These integer invariants hint at a fundamental quantisation of conserved quantities such as charge, spin etc.

5 The Special Relativity Doppler Solution

The first tentative links to Special Relativity were briefly alluded to earlier in Section (4.8) when discussing the Minkowski form of the URM3 **T** operator (4.39).

Consider a particle moving along the *z*-axis, with velocity *v*, i.e.

$$v = \frac{dz}{dt} \text{ or } z = vt, \tag{5.0}$$

then, in URM3, the vector $\mathbf{X}_{3-}$ representing its position *x* at time *t*, proper time τ, i.e. an STR event in one spatial, one time dimension (x, ct), is given by

$$\mathbf{X}_{3-} = \begin{pmatrix} -c\tau \\ z \\ -ct \end{pmatrix}, \tag{5.1}$$

where τ and t are related by the usual STR definitions:

$$t = \gamma\tau,\ \gamma = \frac{1}{\sqrt{\left(1-(v/c)^2\right)}},\ \gamma \geq 1,\ v \leq c. \tag{5.2}$$

The signage of the elements of $\mathbf{X}_{3-}$ (5.1) is intentional and derives from algebraic convenience when solving URMT's five-dimensional, 'URM5' eigenvector solution.

Note that neither *z* nor γ, in the above, are the same as those defined earlier. Symbol *z* usually denotes the third element of $\mathbf{X}_{3+}$ (4.38), and therefore an acceleration under the SPI (4.42), whilst γ is usually the third element of $\mathbf{X}_{3-}$ (4.38). To reiterate, *z* here is now a position coordinate and γ is the dilation factor in Special Relativity [11]. Suffice to say, symbol notation is largely historical in URMT and the reader is requested to temporarily accept the change of notation used here in this section on Relativity.

According to the relation $\mathbf{X}^{3+} = \left(\mathbf{T}\mathbf{X}_{3-}\right)^T$ (4.40) then, using (5.1), the reciprocal vector $\mathbf{X}^{3+}$ is given by

$$\mathbf{X}^{3+} = \begin{pmatrix} -c\tau & z & ct \end{pmatrix}, \tag{5.3}$$

and thus the conservation equation (4.43) becomes

$$\mathbf{X}^{3-}\mathbf{X}_{3+} = (c\tau)^2 + z^2 - (ct)^2 = 0. \tag{5.4}$$

Using $z = vt$ and rearranging gives

$$t^2 = \frac{c^2}{(c^2 - v^2)}\tau^2. \tag{5.6}$$

From the definition of γ (5.2) it is seen here that the lab time and proper time are related by $t = \gamma\tau$, also as per (5.2). Thus, URM3 gives a consistent representation of a single-axis STR event in terms of its eigenvector $\mathbf{X}_{3-}$ under Pythagoras conditions, Section (4.6). However, this is just a taster and, most importantly, URMT develops a relativistic, five-dimensional eigenvector solution with three spatial dimensions plus the two aforementioned temporal dimensions, summarised as follows, see [13] for full details.

The five-dimensional representation of a four-vector, STR event (x, y, z, ct) [11] is given by the URM5 vector $\mathbf{X}_{5-}$

$$\mathbf{X}_{5-} = \begin{pmatrix} -c\tau \\ x \\ y \\ z \\ -ct \end{pmatrix}. \tag{5.7}$$

The reciprocal, row eigenvector $\mathbf{X}^{5+}$ is defined in the usual URMT way (4.40) by

$$\mathbf{X}^{5+} = (\mathbf{T}^5 \mathbf{X}_{5-})^T, \tag{5.8}$$

where the 5×5 URM5 matrix operator $\mathbf{T}^5$ is defined in block matrix form using the 4×4 identity matrix $\mathbf{I}_4$ as follows

$$\mathbf{T}^5 = \begin{pmatrix} \mathbf{I}_4 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.9}$$

so that $\mathbf{X}^{5+}$ becomes, according to (5.8),

$$\mathbf{X}^{5+} = \begin{pmatrix} -c\tau & x & y & z & ct \end{pmatrix}. \tag{5.10}$$

Once again, the $\mathbf{X}_{5-}$ vector has a zero norm in accordance with an invariant, non-zero, STR interval $c\tau$, i.e.

$$\left\| \mathbf{X}_{5-} \right\|^2 = \mathbf{X}^{5+} \mathbf{X}_{5-} = 0. \tag{5.11}$$

which expands to

$$(c\tau)^2 + x^2 + y^2 + z^2 - (ct)^2 = 0, \tag{5.12}$$

For the purposes of the result that follows, it is only necessary to, once, again, consider motion in one-dimension along the *z*-axis, as per the three-dimensional eigenvector $\mathbf{X}_{3-}$ (5.1), and $\mathbf{X}_{5-}$ becomes

$$\mathbf{X}_{5-} = \begin{pmatrix} -c\tau \\ 0 \\ 0 \\ vt \\ -ct \end{pmatrix}, \; x = 0, \; y = 0, \; z = vt. \tag{5.13}$$

By defining the 'Doppler' parameter α as

$$\alpha = \sqrt{\left(\frac{c+v}{c-v}\right)}, \tag{5.14}$$

then the URM5 eigenvector solution $\mathbf{X}_{5-}$ (5.13) is given by

$$\mathbf{X}_{5-} = \frac{ct}{\left(\alpha^2+1\right)}\begin{pmatrix} -2\alpha \\ 0 \\ 0 \\ +\left(\alpha^2-1\right) \\ -\left(\alpha^2+1\right) \end{pmatrix}. \tag{5.15}$$

Although not derived herein, see [13], the full, five-dimensional URM5 eigenvector solution is very similar to that of URM3 (by design) with a 'plus' acceleration vector $\mathbf{X}_{5+}$ and a 'zero' velocity vector $\mathbf{X}_{50C}$ reproduced below from [13] as follows:

$$\mathbf{X}_{5+} = \frac{c}{t}\left(\frac{\alpha^2+1}{\alpha^2}\right)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},\ \mathbf{X}_{50C} = \frac{c}{\alpha}\begin{pmatrix} \alpha \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{5.16}$$

The eigenvalues are $\lambda = \pm C,0,0,0$, i.e. two non-zero eigenvalues $\pm C$, and three zero eigenvalues, with $\mathbf{X}_{5+}$ the eigenvector for $\lambda = C$, $\mathbf{X}_{5-}$ the eigenvector for $\lambda = -C$, and three 'zero' eigenvectors $\mathbf{X}_{50A}$, $\mathbf{X}_{50B}$ and $\mathbf{X}_{50C}$ for the three zero eigenvalues (only $\mathbf{X}_{50C}$ is shown above).

Note that the 5x5 matrix $\mathbf{A}_{50}$, for which these are eigenvectors, is given later (6.1) in a slightly more general form.

Of most interest here is actually the above, invariant acceleration vector $\mathbf{X}_{5+}$, which is seen to be inversely proportional to the time t. With the two spatial coordinates x and y zero (second and third elements of $\mathbf{X}_{5+}$), then the non-zero acceleration a is given by the fourth element, i.e.

$$a = \frac{c}{t}\left(\frac{\alpha^2+1}{\alpha^2}\right). \tag{5.17}$$

From this expression, for speeds close to the speed of light, when α is very large, then the acceleration a is approximated as

$$a \approx \frac{c}{t},\ \alpha >> 1, \tag{5.18}$$

which is just the Hubble-equivalent, expansion acceleration. Evidently, if $t = 0$, the solution blows up (quite literally) with a singularity. Given the age of the universe is approximately 14 billion years, i.e.

$O(10^{10})$ years, which equates to about 10^{17} seconds, and with $c \approx 3\times10^{8}\, m/s$, then the acceleration in this epoch is approximately

$$a \approx O(10^{-9})\ ms^{-2}, \text{ URMT acceleration,} \tag{5.19}$$

which is the acceleration equivalent of the Hubble constant $H = 73.8 \pm 2.4\times10^{3}\ ms^{-1}/Mpc$ (or $2.4\times10^{-18}\ ms^{-1}/m$), where $1Mpc = 3.086\times10^{22}\, m$. This derives as follows: at distance *r*, the recession velocity is given by the Hubble law as $v = Hr$, and differentiating this gives the acceleration $a = \dot{v} = H\dot{r}$. Using $v = \dot{r}$ this becomes $a = Hv$. For velocities near the speed of light $a = \dot{v} \approx Hc$ for $v \approx c$. Substituting for $H = 2.4\times10^{-18}\ ms^{-1}/m$ and $c = 3.0\times10^{8}\ ms^{-1}$ gives $a = 7.2\times10^{-10}\ ms^{-2}$.

The above gives the acceleration in the current epoch. However, at the very earliest, non-zero time, such as the Planck time $t = O(10^{-44})s$, then the acceleration is a huge $O(10^{50})ms^{-2}$.

6 The Special Relativity Mass Solution

6.1 Introduction

The above Doppler solution actually derives from the eigenvectors to the following URM5, 5x5 matrix:

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & M & (\tau/2)\mathbf{X}^{3-} \\ -M & 0 & (t/2)\mathbf{X}^{3-} \\ (-\tau/2)\mathbf{X}_{3+} & (-t/2)\mathbf{X}_{3+} & \mathbf{A}_{30} \end{pmatrix}. \tag{6.1}$$

The characteristic equation for this matrix, i.e. the 5-dimensional equivalent of the Dynamical Conservation Equation (4.5), is

$$0 = \lambda(\lambda^2 - C^2)(\lambda^2 + M^2), \tag{6.2}$$

and the eigenvalues are thus

$$\lambda = \pm C, \pm iM, 0. \tag{6.3}$$

It is noted that if M is zero then the eigenvalues reduce to $\lambda = \pm C, 0, 0, 0$, which is the key simplification made to obtain the Doppler solution, given in the previous section. However, here, the dynamical variable *M* will not be assumed zero, and this is because it is related to mass and the potential term *V* in the DCE (4.6). What this means, therefore, is that for zero *M*, and/or a zero *V*, the earlier Doppler solution and all prior URM3, 3-dimensional Pythagorean solutions, with a naturally occurring zero potential energy, are considered to represent massless particle solutions, i.e. URMT has, so far, provided solutions for the photon or graviton particles. The mathematical justification for linking *M* with mass, and simultaneously also the potential *V*, is given next.

The eigenvalue equation is factored as follows, disregarding (or factoring) the eigenvalue λ for the zero eigenvalue:

$$\lambda^4 = (C^2 - M^2)\lambda^2 + M^2C^2. \tag{6.4}$$

Defining the kinetic term as

$$K = C^2 - M^2 . \tag{6.5}$$

and the potential V term as

$$V = M^2 . \tag{6.6}$$

then (5.10) becomes

$$\lambda^4 = K\lambda^2 + VC^2 . \tag{6.7}$$

This is a quartic with quadratic and squared quadratic terms only and will, very shortly, be related to the relativistic energy momentum equation. Beforehand, for eigenvalues $\lambda = \pm C$, the characteristic equation (6.7) now becomes

$$C^4 = KC^2 + VC^2, \ \lambda = \pm C \tag{6.8}$$

and dividing throughout by C^2, which is always greater than zero by (4.0), returns the familiar DCE:

$$C^2 = K + V . \tag{6.9}$$

and justifies the revised kinetic and potential definitions, (6.5) and (6.6) respectively.

The non-zero, dynamical variable M in $\mathbf{A}_{50}$ is very important in URMT due to its association with a non-zero rest mass and, in keeping with Relativity, it therefore possesses a sub-luminal speed and is known as the 'reduced velocity', Appendix (B), since it can actually be interpreted as a velocity down from the speed of light, instead of growing from 0 to just below the speed of light.

Note that M itself is actually a velocity quantity, and not a mass. In fact, it is more useful to think of it as a momentum per unit mass. The reduced velocity is also expanded upon further in Appendix (B), but attention now turns to comparing the DCE (6.7) with the relativistic energy-momentum.

6.2 The Relativistic Energy-momentum Equation

Throughout URMT, starting right at the beginning with URM3 [2], and extending to URM4 and URM5 in [14], the characteristic equation, i.e. the DCE, is considered an energy conservation equation (per unit mass), with a total energy given by the invariant eigenvalue C^2. The eigenvalue C is inevitably associated with the speed of light c (4.7b) hence the familiar look to the relativistic energy formula $E = K = C^2 = c^2$ (per unit mass) for a zero potential, i.e. $V = 0$ in (4.7). Much of URMT, particularly in [14], concentrates on zero potential energy solutions and, as such, all the energy is kinetic. In essence, all very much like a particle with a zero rest mass but, nevertheless, finite energy C^2 (per unit mass). This is why, repeatedly, the invariant, zero potential solution, Section (4.7), is considered primarily a massless particle solution, i.e. a photon or graviton. The introduction of a non-zero dynamical variable M now changes all that.

Returning to the characteristic equation (6.8), it is noted to be fourth order in C. Furthermore, given it splits nicely into two terms, i.e. a kinetic and potential term, it can be directly compared with the STR relativistic energy-momentum equation

$$E^2 = p^2c^2 + {E_0}^2, \tag{6.10}$$

where, as usual, p is the momentum of an object (particle) with relativistic mass m and velocity v, i.e. $p = mv$, and the total energy E is given by Einstein's equation:

$$E = mc^2, \tag{6.11}$$

with the rest mass (m_0) energy E_0 given by:

$$E_0 = m_0 c^2. \tag{6.12}$$

Expanding (6.10) in component form gives

$$(mc^2)^2 = (mv)^2 c^2 + (m_0 c^2)^2, \tag{6.13}$$

and dividing throughout by the squared mass m^2 gives

$$c^4 = v^2 c^2 + \left(\frac{m_0}{m}\right)^2 c^4. \tag{6.14}$$

Comparing this with the characteristic equation (6.8) gives the following associations of the energy terms, all, strictly speaking, per unit mass:

$$\begin{aligned} E = C^2 &= c^2, \text{ per unit mass,} \\ K = C^2 - M^2 &= v^2, \text{ ditto,} \\ V = M^2 &= \left(\frac{m_0}{m}\right)^2 c^2, \text{ ditto.} \end{aligned} \tag{6.15}$$

From STR [11], the ratio of masses m_0/m is equal to the reciprocal of γ (5.2), i.e.

$$\frac{1}{\gamma} = \frac{m_0}{m}, \tag{6.16a}$$

and by substituting this ratio into the potential term (6.15), the dynamical variable M is seen to be related to the eigenvalue (now also the speed of light) by

$$C = \gamma M, \tag{6.16b}$$

which is actually its original definition as a 'reduced velocity', see Appendix (B).

6.3 The URMT Rest-mass Energy Equation

Lastly, and most importantly, using $C = \gamma M$ (6.16b) and $C = c$ (4.7b), then the rest mass energy E_0 (6.12), becomes

$$E_0 = m_0 \gamma M C, \tag{6.17}$$

and since $m = m_0\gamma$ by (6.16a) then the rest mass energy can be written in terms of the URMT dynamical variable M and eigenvalue C as

$$E_0 = mMC \text{, URMT's rest mass energy equation.} \tag{6.18}$$

Superficially, this is just a rewrite of $E_0 = m_0c^2$, with the rest mass term m_0c effectively replaced by mM, but what was the STR set $\{m_0, m, v, c\}$, of two masses and two velocities, is now replaced by a single mass and three velocities, $\{m_0, M, v, C\}$, where the single, relativistic mass m now cancels across the energy-momentum equation (6.13) such that URMT's equivalent equation becomes the Pythagorean relation given earlier (5.4), rearranged and written in terms of the eigenvalue C as

$$C^2 = M^2 + v^2 \text{, URMT's energy-momentum equation.} \tag{6.19}$$

Whilst it may be argued that both (6.18) and (6.19) are just rewrites of STR's equivalent energy equations, the reader is reminded that the URMT, invariant eigenvalue C originates from URM3 and a number-theoretic problem in linear algebra. Furthermore, both energy equations, (6.18) and (6.19) are now symmetric, to within a sign, upon interchange of the velocities, i.e. M and C in $E_0 = mMC$, and M, v and C in $C^2 = M^2 + v^2$. In particular, this now puts the abstract, reduced velocity M on an equal footing with the familiar, laboratory velocity v. The addition of M is seemingly just an extension of the URM5 Doppler solution, yet it appears directly related to rest mass and also the potential energy (6.6).

It would seem, therefore, that by starting bottom-up with a problem in number theory, i.e. obtaining the integer eigenvalues and eigenvalues of the matrix $\mathbf{A}_{50}$ (6.1) according to an invariance principle, Section (4.3), the STR energy equations can be derived. Indeed, although not shown here, URMT can also relate a unity root matrix to both an event and its Lorentz transform [13].

7 Higher Dimensional Compactification and Holography

With URM3 representing a three dimensional theory, the higher-dimensional extensions to a four and five dimensional representation are given by URM4 and URM5, using 4x4 and 5x5 matrices respectively – an example of URM5 is given by the Doppler solution in Section (5). These higher dimensions can be shown to compactify, i.e. become small with respect to the first three dimensions by parametric variation. Here, under the SPI, this parametric variation is simply the passage of time as represented by the temporal parameters t_4 and t_5, one per dimension (fourth and fifth), as demonstrated next.

The matrix $\mathbf{A}_{50}$ (5.5), for zero *M*, is actually a variant of the following, more general matrix, where t_4 and t_5 are arbitrary temporal parameters for the fourth and fifth dimensions, analogous to the URM3 parametric time t_3 for the third dimension (4.53):

$$\mathbf{A}_{50} = \begin{pmatrix} 0 & 0 & -t_5\mathbf{X}^{3-} \\ 0 & 0 & -t_4\mathbf{X}^{3-} \\ t_5\mathbf{X}_{3+} & t_4\mathbf{X}_{3+} & \mathbf{A}_{30} \end{pmatrix} \tag{7.1}$$

$$t_4,\ t_5 \in \mathbb{Z},\ units(t_4, t_5) = T \text{, time} \tag{7.1b}$$

Comparison with $\mathbf{A}_{50}$ (5.5) shows that the Doppler solution assigns t_4 and t_5 to scaled forms of the familiar laboratory time t and proper time τ, i.e. $\tau = -2t_5$, $t = -2t_4$ but this is largely inconsequential here and t_4 and t_5 are just arbitrary temporal parameters that are not necessarily the relativistic times t and τ.

In fact, for the purposes of illustrating geometric compactification, it is the more general solution for the five-dimensional, 'position' vector $\mathbf{X}_{5-}$ that is of most interest, and is given below in terms of the URM3 eigenvectors,

$$\mathbf{X}_{5-} = \begin{pmatrix} 2Ct_5 \\ 2Ct_4 \\ \mathbf{X}_{3-} - (t_5^2 + t_4^2)\mathbf{X}_{3+} \end{pmatrix}. \tag{7.2}$$

The first two elements $2Ct_4$ and $2Ct_5$ represent the position in the fourth and fifth spatial dimensions, and the position in the first three dimensions is given in terms of the URM3 eigenvector solution by the vector $\mathbf{X}_{3-} - (t_5^2 + t_4^2)\mathbf{X}_{3+}$.

Denoting this position vector by the five spatial coordinates x_1 to x_5, i.e.

$$\mathbf{X}_{5-} = (x_1, x_2, x_3, x_4, x_5), \tag{7.3}$$

then the individual coordinates are

$$(x_1, x_2, x_3) = \mathbf{X}_{3-} - (t_5^2 + t_4^2)\mathbf{X}_{3+}, \tag{7.4}$$

$$x_4 = 2Ct_4, \tag{7.5}$$

$$x_5 = 2Ct_5, \tag{7.6}$$

It is easily seen from the above that the fourth and fifth dimensions are linear functions of time, whilst the first three dimensions are quadratic. Thus, as either of the times t_4 and/or t_5 grow large, the first three dimensions dominate the eigenvector solution and actually align with the URM3 'acceleration' vector $\mathbf{X}_{3+}$, i.e.

$$\mathbf{X}_{5-} \approx \begin{pmatrix} 0 \\ 0 \\ -(t_5^2 + t_4^2)\mathbf{X}_{3+} \end{pmatrix},\ t_4 >> 0,\ t_5 >> 0. \tag{7.7}$$

In other words, for large times, t_4 and t_5, the fourth and fifth dimension shrink (compactify) with respect to the first three dimensions hence URMT exhibits the geometric property of compactification of its multi-dimensional, discrete eigenvector solution.

Note that the eigenvector $\mathbf{X}_{3+}$ itself is Pythagorean throughout most of this paper, and is therefore strictly two-dimensional in so far as its solution space is a 'surface' or, more accurately, a set of points lying on a discrete conical surface as given by the upper cone $\mathbf{C}_U$, Section (4.14). This discrete surface is parameterised by integers *k* and *l* in the parametric solution (4.51), hence a two-dimensional

surface. Nevertheless, the complete eigenvector solution is three-dimensional, since the extra parameter is, in fact, the temporal parameter t_3 in (4.53). However, it is actually URM4, i.e. a 'four-dimensional' solution that can only truly represent three-dimensional, arbitrary vectors in URMT, under the methods of 'Arbitrary Vector Embedding' - see [13] and [15]. And, as per the Doppler solution, a four-dimensional relativistic event is treated under the five-dimensional scheme that is URM5 [13]. In brief, URMT solutions add an extra dimension to the physical problem and essentially then projects down to the 'surface' that is one dimension less, i.e. URMT solutions have a holographic nature to them – the physics is mathematically formulated one dimension higher than the physical manifestation.

8 Summary

URMT originates from the pure mathematical branch of number theory and Diophantine equations (integer equations), with a modified variant of the FLT equation, modified such that it gives an infinite set of solutions, each an eigenvector to a unity root matrix, i.e. a matrix whose elements are unity (or primitive roots), for a unity eigenvalue. The non-singular condition on the unity root matrix leads to the characteristic eigenvalue equation, interpreted as an energy conservation equation, akin to a Hamiltonian. Indeed, upon extension to higher dimensions, it is seen to be equivalent to the relativistic energy-momentum equation, with the invariant, unity eigenvalue none other than the speed of light.

The energy conservation equation is then be solved by variational methods to give both the dynamical equations (the eigenvector equations) and their solution, i.e. the eigenvectors, that are given in terms of the elements of the matrix (the unity roots), also interpreted as rate quantities, e.g. velocities, and consequently relabelled as dynamical variables.

Reducing the original Diophantine equation to a quadratic exponent, and under simplifying Pythagoras conditions, the three eigenvectors can be associated with acceleration, velocity and position vectors under a standard physical interpretation of URMT. The eigenvector solution additionally evolves with time, where time itself is a variational parameter.

The eigenvector inner products are conservation equations and, for a unity eigenvalue, give the three most basic scalars {0,1,2}, all of which are invariant to temporal evolution (parametric transformation) of the eigenvectors, giving the transformations a unitary symmetry as per QM. This then leads to a modified 'Quantum Physical Interpretation', whereby the energy quantities are replaced by the action. This then leads to a second interpretation of the invariant eigenvalue to Planck's constant. Thus, URMT has related its invariant eigenvalue to two physical constants, c and h, but with same mathematics. Although not demonstrated herein, URMT's discrete eigenvector lattice solution shows inverse square law curvature with the eigenvalue tentatively thus also related to the gravitational constant 'big G', see [2]#3.

An example, five-dimensional 'Doppler solution' shows a huge initial acceleration at the earliest instance of time, decaying in accordance with the Hubble expansion law. A second example shows how relativistic mass can be introduced implicitly into URMT by addition of a single, new dynamical variable, i.e. the 'reduced velocity'. This leads directly to the relativistic energy-momentum equation and a reformulation of the rest mass energy in a symmetric form involving both the reduced velocity and the speed of light. Furthermore, temporal evolution of the higher dimensions leads to the geometric property of compactification, where the higher dimensions appear to shrink relative to the lower dimensions over long evolutionary periods, and highlights URMT's holographic nature, i.e. physical solutions are actually constructed in one dimension higher than they manifest themselves in reality.

To keep the paper relatively short, some other important physical associations have had to be omitted. Nevertheless, it should be mentioned that URMT also has a six-dimensional representation of the

Quark Flavour Model, stemming from URMT's inherent possession of unitary symmetry. This also leads to a three-fold increase in its solution space, and with this a straightforward explanation of three-fold quark colour degeneracy [7].

9 Conclusions

What originated as a study in the residue properties of a modified form of Fermat's Last Theorem, and the resulting eigenvector solution in terms of unity roots, in particular with regard to their symmetry and invariants, appears to generate a surprisingly rich field of physical phenomena, and supports the author's premise that, at the smallest, Planck scale, nature reduces to some very simple rules, with its laws formulated as integer equations, and more the realm of number theory than physics. The laws of nature are thus reduced to the legal combinations of integers, which currently appear to be of quadratic degree, in particular hyperbolic and n-dimensional Pythagoras.

10 References

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This book is broken into six separate papers, each paper is given a specific reference #1 to #6 as follows:

[2]#1 Unity Root Matrix Theory Foundations
[2]#2 Pythagorean Triples as Eigenvectors and Related Invariants
[2]#3 Geometric and Physical Aspects
[2]#4 Solving Unity Root Matrix Theory
[2]#5 Unifying Concepts
[2]#6 A Non-unity Eigenvalue

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11 Appendices

11.1 Appendix (A) Unity Roots

11.1.1 What is a Unity Root?

Most readers will be familiar with the complex, nth roots of unity as given by the solutions to the polynomial

$$z^n = 1. \tag{A1}$$

For the quadratic exponent, $n = 2$, there are the two, real roots $z = \pm 1$. For general, non-zero exponent n, there are *n* unique roots, and all are complex for $n > 2$ with their complex, polar form given by

$$z = e^{2\pi k i / n} \;\; \forall k \in \mathbb{Z},\; n \geq 1. \tag{A2}$$

It is easily verified that this is the general solution since

$$z^n = e^{(2\pi k i / n)n} = e^{2\pi k i} = 1 \tag{A3}$$

Unlike the above, complex-valued roots, URMT is entirely based in integers and, in the field of number theory, there are integer equivalents to the above, but this also introduces an additional modulus in the process with subsequent modulo arithmetic, which then becomes the realm of congruences in number theory [16]. These integer roots are known as primitive (or unity) roots, and are a much-studied area in classical number theory. For example, a unity root '*P*', exponent *n*, modulus *x*, is defined by the congruence relation (explained next)

$$P^n \equiv 1 \pmod{x}, \;\; \forall x \in \mathbb{Z},\; |x| \geq 2. \tag{A4}$$

Note that the modulus *x* in (A4) can also be negative, but cannot meaningfully be zero or plus or minus one.

In other words, when P is raised to the power *n*, its remainder, when divided by integer modulus *x*, is unity. For example, in the cubic exponent case, $n = 3$, modulus $x = 7$, one of three unity roots is $P = 2$ so that

$$2^3 \equiv 1 \pmod{7}. \tag{A5}$$

This expands trivially to $2^3 = 1.7 + 1$, hence 2^3 gives a remainder of one when divided by seven. Note that, in number theory, the remainder is referred to as a residue and, given the power in this example is cubic, then the above equation is alternatively described as 'one is a cubic residue of two, modulo 7'.

For this cubic exponent example there are also three unique unity roots, as per a real-valued, cubic polynomial, and these are simply 1 and 2

$$1^3 \equiv 1 \pmod{7} \tag{A6a}$$

$$2^3 \equiv 1 \pmod{7}. \tag{A6b}$$

$$4^3 \equiv 1 \pmod 7 . \tag{A6c}$$

It is noted that unity is trivially a root for all exponents and modulus *x*, as per the real-valued equivalent, i.e. $1^n \equiv 1 \pmod x$ for all $|x| \geq 2$.

Thus, there is always at least one unity trivial root '1'. More generally, in accordance with Lagrange's theorem [16], for prime modulus *x*, there are always *n* distinct roots if, for arbitrary integer *f*, the modulus is of the form $fn+1$. In fact, neither the modulus nor the exponent have to be prime and, in fact, there may be more than *n* roots. However, for the purposes of this paper, it makes life simpler to keep with both prime moduli and exponents. Note that in the complex polynomial case of the nth degree (A1), there are always *n* unique roots, regardless of the form of the exponent *n*, i.e. be it prime, composite and or of the $fn+1$ form, as per a complex polynomial of the nth degree – this fact is known as the *fundamental theorem of algebra*, see [3].

Furthermore, there is no restriction that the modulus necessarily be positive, i.e. *x* could be negative, actually $x \leq -2$, since a modulus of absolute, unity value always has a zero residue, i.e. all residues modulo 1 or -1 are zero since 1 and -1 trivially divide every integer leaving no 9zero) remainder

Also note that if *P* is a root then, for arbitrary integer *m*, so is $P+mx$ since

$$(P+mx)^n \equiv P^n \pmod x . \tag{A9}$$

In the example above, since 2 is a root, so too therefore is 9 ($=2+1.7$) and 16 ($=2+2.7$) etc., i.e. if $2^3 \equiv 1 \pmod 7$ then $9^3 \equiv 1 \pmod 7$ and $16^3 \equiv 1 \pmod 7$. However, it is only necessary to restrict the roots *P* to the principle range, which is 1 to $x-1$ here, i.e. all values *P* modulo *x*, $P \geq 1$ here for unity roots.

This is really no different to the general complex case where, if *z* is a unity root (A2), so too is $z = e^{2\pi(k+mn)i/n}$ for arbitrary integer *m*, but only those in the range $0 \leq k < n$ need be considered.

For prime exponent *n*, just as in the complex case, if *z* is a root (except the trivial root '1'), then so too is z^2, z^3 etc, and any root *z* forms a 'generator' of the cyclic group (order *n*) of roots [12]. In the above cubic exponent example, 2 is a root, and so too is 2^2, i.e. 4 since $4^3 \equiv 1 \pmod 7$.

On the other hand, whilst 2^3 is also a valid root, it is just a repeat of the smallest unity root (1) since $2^3 \equiv 1 \pmod 7$. Likewise, 2^4 repeats the root 2 since $2^4 \equiv 2 \pmod 7$, and 2^5 repeats the root 4' since $2^5 \equiv 4 \pmod 7$. Thus, there are really only three distinct cubic roots modulo 7, i.e. the set {1,2,4}, and the three roots can be obtained from the generator 2. This is also the case using 4 instead of 2, i.e. 4, 4^2 and 4^3 which evaluate to 4, 2 and 1, modulo 7, respectively , i.e. the same set of roots {1,2,4}, just shuffled in a different order.

Indeed, the complex roots of unity also form a cyclic group and, for prime exponent, any root (except '1') can act as a generator of all the other $n-1$ distinct roots. In other words, only a single root is required to be able to determine all other roots. For composite exponents, the roots still form cyclic groups, but these will have 'proper' sub-groups, i.e. sub-groups of order smaller than *n*, whereas cyclic roots for a prime exponent have no proper sub-groups.

In the above example, the exponent *n* is prime (3 is prime) and the modulus 7 is also prime. It should be noted that the modulus relates to the exponent by the simple arithmetic $7 = 2.3 + 1$, which is of the $fn + 1$ form mentioned further above, where $f = 2$ here, and thus there are three roots by Lagrange's Theorem. In fact, URMT literature usually states the following form:

$$x = f2n + 1, \tag{A10}$$

where *f* would now be just one using the numbers above. This latter '*f2n*' form is actually pertinent to odd, prime exponents.

Whilst unity root *P* has been used in the example above, the URMT root *Q* is also defined similarly to *P*, but its modulus is coordinate *y* i.e.

$$Q^n \equiv 1 \pmod{y}. \tag{A11}$$

There is also the URMT root *R*, which is actually defined as a root of minus one, modulo the coordinate *z*, i.e.

$$R^n \equiv -1 \pmod{z}. \tag{A12}$$

This last twist, i.e. a change of root from one to minus one, is basically the URMT equivalent of nth degree complex roots of -1, i.e.

$$z^n = -1. \tag{A13}$$

When using integers and modulo arithmetic, the negative root -1 is equivalent to $z - 1$ modulo *z*, since $z - 1 \equiv -1 \pmod{z}$ so that, alternatively,

$$R^n \equiv z - 1 \pmod{z}. \tag{A15}$$

In the cubic, modulo 7 example above, in addition to the three roots of unity +1, i.e. the set {1,2,4}, there are also three roots of -1. These are the set {3,5,6}, e.g.

$$3^3 \equiv 6 \pmod{7} \text{ since } 6 \equiv -1 \pmod{7}, \text{ i.e. } 6 = 1.7 + (-1), \tag{A16}$$

and similarly $5^3 \equiv 6 \pmod{7}$ and $6^3 \equiv 6 \pmod{7}$.

Unlike the positive root case, the two non-unity values 3 or 5 do not as generators of the set but they can be used in combination with a positive root. For example, taking the positive root 2, and successively multiplying the negative root 3 by it, gives 6 and 12, i.e. -1 and 5 modulo 7, both of which are roots of -1, thus forming the set {3,6,5}. Likewise for negative root 5, successively multiplying it by the positive root 2 gives 10 (=3 modulo 7) and 20 (=6 modulo 7) and the same set of roots of -1, reshuffled as in {5,3,6}.

Thus, taking into account both positive and negative roots, there are two sets of three roots in this example, i.e. six roots in all, {1,2,4} and {3,5,6}. Given the principle range of residues modulo 7 is the seven integer values 0 to 6, only zero is missing. Of course, raising zero to any non-zero power gives zero, so it can never be a unity root. Whilst obvious, the point here is that the zero value is accounted for by the '+1' in the form $f2n + 1$ (A10), and the '*2n*' accounts for the two sets of three, for the cubic exponent, which makes $f = 1$. The total $2f + 1$ is therefore 7 i.e. the same as the modulus. Note that the next prime modulus of $f2n + 1$ form, for $n = 3$, is 13, and *f* is now 2. In this case there

are still only three positive unity roots and three negative unity roots (three in each case because the exponent remain cubic), thus six roots in total, and with a principal range, modulo 13, of thirteen integer values 0 to 12, there remain 7 values that are not unity roots - and you can therefore forget about them. This last point meaning that, in general, for arbitrary prime modulus, the exponent is usually much smaller than the modulus and most values in the principal range are not unity roots.

11.1.2 Conjugate Roots

In the complex case (A2), the unity roots have conjugate forms given by

$$z^* = e^{-2\pi ki/n}, \tag{A17}$$

which are also unity roots, i.e.

$$\left(z^*\right)^n = 1, \tag{A18}$$

and in URMT, similarly, the unity roots P,Q,R have conjugate forms $\overline{P},\overline{Q},\overline{R}$ that also satisfy the same unity root properties as per their standard forms P,Q,R, i.e.

$$\overline{P}^n \equiv 1 \pmod{x}, \tag{A19}$$

$$\overline{Q}^n \equiv 1 \pmod{y}, \tag{A20}$$

$$\overline{R}^n \equiv -1 \pmod{z}. \tag{A21}$$

In addition, they relate to their standard forms by the following 'conjugate relations'

$$\overline{P} \equiv P^{n-1} \pmod{x},\ \overline{Q} \equiv Q^{n-1} \pmod{y},\ \overline{R} \equiv -R^{n-1} \pmod{z} \tag{A22}$$

From these definitions, it is easily seen that the products $P\overline{P}$, $Q\overline{Q}$ and $R\overline{R}$ all equate to unity, i.e.

$$P\overline{P} \equiv P^n \equiv 1 \pmod{x},\ Q\overline{Q} \equiv Q^n \equiv 1 \pmod{y},\ R\overline{R} \equiv -R^n \equiv 1 \pmod{z}. \tag{A23}$$

which is also as per the complex case, i.e.

$$zz^* = e^0 = 1. \tag{A24}$$

In both complex and URMT cases, the roots conjugated twice return the same value

$$\left(z^*\right)^* = z,\ \overline{\overline{P}} \equiv P \pmod{x} \text{ etc. for } Q \text{ and } R. \tag{A25}$$

The concept of conjugates extends wider in Physics to, for example, the representation of particles and their conjugate anti-particles [7] or, more generally, wave-functions and their Hermitian conjugate forms in QM[8].

11.2 Appendix (B) The Reduced Velocity M

In URMT, a very useful quantity, termed the 'reduced-velocity', and denoted by the symbol M, is defined as follows in terms of the speed of light *c*, and the dilation factor γ (5.2):

$$M = \frac{c}{\gamma},\ 0 \le M \le c \text{ for } \gamma \ge 1. \tag{B1}$$

It is seen that with $\gamma > 1$ then M is less than the speed of light.

The following two limiting cases are of note, derived from the definition of γ (5.2),

$$v = c \ \Rightarrow\ \gamma = \infty,\ M = 0, \tag{B2}$$

$$v = 0 \ \Rightarrow\ \gamma = 1,\ M = c. \tag{B3}$$

For a massless particle, e.g. a photon or graviton with a velocity that of the speed of light, then case (B2) applies and M is zero. For a particle at rest, with zero velocity, then case (B3) applies and M is the speed of light c.

With these points in mind, M is termed 'reduced' because it is zero at the speed of light and grows to the speed of light as the speed v decreases to zero, i.e. it is the speed reduced from that of c, whereas v increases from 0 to c. When M is greater than zero it is considered equivalent (but not numerically the same) to the speed v of a particle with finite mass and sub-luminal velocity, i.e. $v < c$.

From M (B1) and γ (5.2) the following important relation between c, M and v is obtained

$$0 = M^2 + v^2 - c^2, \tag{B4}$$

which rearranges to give an alternative form for defining M as

$$M = \sqrt{c^2 - v^2}, \tag{B5}$$

where the positive root is taken for positive γ and c.

The relationship (B4) is yet another Pythagoras equation, and there seems to be no escaping this simple equation throughout URMT physics.

What makes M extra special is that it is also a dynamical variable in the URM5 unity root matrix $\mathbf{A}_{50}$ (6.1), and not just an ad-hoc definition introduced for algebraic or physical convenience.