A Quark Flavour Model in Integers
(Pythagorean Sextuplets as Quarks and Related Invariants)

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Abstract

This paper gives an integer-based, number-theoretic representation of the quark flavour model. The six known quarks, as per a standard representation, are given by eigenvectors to a matrix operator but, unlike a standard representation, satisfy Pythagorean or hyperbolic Diophantine equations, and are eigenvectors to an all-integer unity root matrix. Comparison of the commutation relations with those of quantum mechanical angular momentum provide an expansion of the single-axis representation to a three-axis, three-fold solution, linked to the red, green and blue quark colours, with the additional three-fold degeneracy in quark generations given by a full 6x6 matrix representation. Parametric evolution of the eigenvector solution is shown to preserve all inner products and thus maintain unitarity as per the special unitary groups SU(n), and also shows that the six-quark solution can geometrically compactify to look like a single quark. Comparison of the time-domain eigenvector evolution with the time-evolution of the wavefunction leads to an interpretation of the parametric variation in the unity root matrix as a form of action principle. The paper ends with a numeric example that can be used to verify the quark algebra.

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Acronyms and Abbreviations

DCE : Dynamical Conservation Equation
QCD Quantum Chromodynamics
QFT : Quantum Field Theory
QM : Quantum Mechanics
QPI : Quantum Physical Interpretation
RGB : Red, Green, Blue
SPI : Standard Physical Interpretation
SU(n) : Special Unitary group, order n
URM n : the $n \times n$ matrix formulation of URMT
URMT : Unity Root Matrix Theory

Contents

1 Background

This paper is essentially a summary of the key points in Reference [1] and therefore concentrates on those areas where Unity Root Matrix Theory (URMT) has something unique to say, leaving more general concepts such as particle multiplets, unitary transforms and spin for the reader to explore at length in standard particle and quantum physics texts such as [2], [3], [4] and [5] or, indeed, [1]. Nevertheless, it has been attempted to keep the work self-contained so that dipping in and out of references is hopefully minimal –there is plenty of free, background material on URMT at the web site [6], e.g. see the Overview [7].

The readership level has been kept at undergraduate, with explicit usage of Lie algebra and associated symmetry groups SU(n) avoided as unnecessary to illustrate the main points. As a consequence, the work should be amenable to under-graduates having taken a first course in quantum mechanics and/or particle physics (quarks, spin, isospin), [3] and [4], plus some basic linear algebra (matrices, eigenvalues and eigenvectors). Whilst there are many good texts on QM, [2] and [5] are recommended for most if not all of the QM topics covered. Texts [8] and [9] on Quantum Field Theory (QFT) are generally post-graduate level but cover SU(n) and quarks, and provide a more informative background for the advanced student.

Knowledge of number theory is kept to an absolute minimum, with only a very basic knowledge of congruences required and, failing that, just an appreciation that the work is entirely in integers should be sufficient – see [9] for a basic undergraduate text on the subject. On the other hand, the pure number theorist could actually gloss over the physics because, once again, URMT has much to add on the subject of Pythagorean n-tuples and hyperbolic Diophantine equations with regard to parametric variation. In particular, there is much new URMT material including, most importantly, a three-fold expansion of URMTs three-dimensional solution, additionally extended to six-dimensions. As a consequence, this paper could well have been entitled '*Pythagorean Sextuplets as Quarks and Related Invariants*' since it is an advancement of one of the very earliest URMT papers on '*Pythagorean Triples as Eigenvectors and Related Invariants*' [10], which is also a freely available PDF.

2 URM3, Angular Momentum and Quark Isospin

This first section gives an overview of the original 3x3 matrix incarnation of URMT, followed by a comparison of its operator matrices to the angular momentum matrices of quantum mechanics (QM), using their commutation relations and raising and lowering properties. This then prepares the ground for a look at the three quarks, up, down and strange, together with their isospin properties. The comparison with a three-state angular momentum scheme leads, in the next section, to an expansion of URMT's fundamental three operator matrices, $\mathbf{A}_0$, $\mathbf{A}_+$ and $\mathbf{A}_-$ to nine matrices, thereby achieving a full, three-axis, URMT representation of angular momentum, spin and hence also isospin. The founding unity root matrix $\mathbf{A}_0$ is associated with the generator of rotations about a chosen axis, whilst $\mathbf{A}_+$ and $\mathbf{A}_-$ act as the raising and lowering operators for that axis.

The URM3 Eigenvector Solution

The complete URM3, integer, eigenvector solution, under simplifying Pythagoras conditions [7], comprises three $\mathbf{A}$ matrices, $\mathbf{A}_+$, $\mathbf{A}_0$ and $\mathbf{A}_-$, and three associated eigenvectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$. These matrices and eigenvectors are defined in terms of three sets of integer variables, x, y, z , P, Q, R and α, β, γ as follows:

(2.1)

$$\mathbf{A}_+ = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ -y & x & 0 \end{pmatrix},\ x, y, z \in \mathbb{Z},\ (x, y, z) \neq (0,0,0)$$

$$\mathbf{A}_0 = \begin{pmatrix} 0 & R & Q \\ -R & 0 & P \\ Q & P & 0 \end{pmatrix},\ P, Q, R \in \mathbb{Z},\ (P, Q, R) \neq (0,0,0)\text{, the unity root matrix}$$

$$\mathbf{A}_- = \begin{pmatrix} 0 & -\gamma & -\beta \\ \gamma & 0 & \alpha \\ -\beta & \alpha & 0 \end{pmatrix}\ (\alpha, \beta, \gamma) \neq (0,0,0)$$

(2.2) $\mathbf{X}_+ = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ \mathbf{X}_0 = \begin{pmatrix} P \\ -Q \\ R \end{pmatrix},\ \mathbf{X}_- = \begin{pmatrix} \alpha \\ \beta \\ -\gamma \end{pmatrix}.$

The x, y, z are known in URMT as 'coordinates', the P, Q, R as 'dynamical variables', and the α, β, γ as 'scale' or 'divisibility factors' [7]. The coordinates and scale factors satisfy Pythagoras, i.e.

(2.3) $0 = x^2 + y^2 - z^2$, $0 = \alpha^2 + \beta^2 - \gamma^2$,

and the dynamical variables P,Q,R satisfy the following hyperbolic, 'Dynamical Conservation Equation' (DCE), for integer eigenvalue C:

(2.4) $P^2+Q^2-R^2=C^2,\ C\in\mathbb{Z},\ C\geq 1$, the DCE.

This hyperbolic equation is a very special equation in URMT physics as its four-dimensional variant can be directly equated with the relativistic energy momentum equation [11], i.e. eigenvalue C is equated to the speed of light 'little *c*'. In fact, the eigenvalue will also later be related to Planck's constant $\bar{h}$.

The dynamical variables P,Q,R satisfy the following congruences:

(2.5) $P^2\equiv C^2 \pmod{x},\ Q^2\equiv C^2 \pmod{y},\ R^2\equiv -C^2 \pmod{z}$,

and expand as follows, which also forms the definition of the 'divisibility factors' α,β,γ in terms of the coordinates x,y,z:

(2.6) $C^2-P^2=\alpha x,\ C^2-Q^2=\beta y,\ C^2+R^2=\gamma z$.

When the eigenvalue magnitude is unity, i.e. $C=1$, the dynamical variables are unity or 'primitive' roots [9], otherwise known as quadratic power residues for non-unity, integer *C*. Because of this unity root property of the dynamical variables, the matrix $\mathbf{A}_0$ (2.1) is known as the 'unity root matrix', and is thus the founding matrix of the subject. The three vectors $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ are eigenvectors of this unity root matrix for the three, distinct eigenvalues $C,0,-C$ respectively, i.e.

(2.7) $\mathbf{A}_0\mathbf{X}_+=C\mathbf{X}_+,\ \mathbf{A}_0\mathbf{X}_0=0,\ \mathbf{A}_0\mathbf{X}_-=-C\mathbf{X}_-,\ \lambda=C,0,-C$.

These equations are known as the 'dynamical equations' in URMT [1], and are the first evidential link to a three-state, quark scheme of up ($\mathbf{X}_+$), strange ($\mathbf{X}_0$) and down ($\mathbf{X}_-$), when matching their eigenvalues $C,0,-C$ with isospins of 1/2,0,-1/2 respectively.

The other two matrices $\mathbf{A}_+$ and $\mathbf{A}_-$ each have only a single, zero eigenvalue, multiplicity three, with the single eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ respectively, i.e.

(2.9) $\mathbf{A}_+\mathbf{X}_+=0,\ \mathbf{A}_+^3=0, \mathbf{A}_-\mathbf{X}_-=0,\ \mathbf{A}_-^3=0$.

Nevertheless, they also have a raising, lowering effect as given by the following equations in the eigenvectors

(2.10)
$\mathbf{A}_+\mathbf{X}_-=2C\mathbf{X}_0, \mathbf{A}_+\mathbf{X}_0=-C\mathbf{X}_+$
$\mathbf{A}_-\mathbf{X}_+=-2C\mathbf{X}_0,\ \mathbf{A}_-\mathbf{X}_0=C\mathbf{X}_-$.

It is seen from these equations that $\mathbf{A}_+$ raises eigenvector $\mathbf{X}_-$ (eigenvalue $-C$ to $\mathbf{A}_0$) to $\mathbf{X}_0$ (eigenvalue 0 to $\mathbf{A}_0$), and then to $\mathbf{X}_+$ (eigenvalue $+C$ to $\mathbf{A}_0$), and finally to zero (2.9). Similarly the lowering operator matrix $\mathbf{A}_-$ lowers eigenvector $\mathbf{X}_+$ to $\mathbf{X}_0$, and then to $\mathbf{X}_-$, and finally to zero (2.9). For any arbitrary value of C, the raising and lowering of the eigenvalue is also of unit C.

This raising and lowering behaviour thus steps the two eigenvectors, $\mathbf{X}_+$ and $\mathbf{X}_-$, through three states, i.e.

(2.11) $\mathbf{A}_+ : \mathbf{X}_- \to \mathbf{X}_0 \to \mathbf{X}_+ \to 0\,, \quad \mathbf{A}_- : \mathbf{X}_+ \to \mathbf{X}_0 \to \mathbf{X}_- \to 0\,.$

Commutators and Angular Momentum

The aforementioned raising and lowering behaviour is identical, barring a scale factor of two in (2.10), to a three-state, angular momentum system in QM [5], with eigenvalues $\bar{h},0,-\bar{h}$, c.f. URMT's eigenvalues $C,0,-C$ (2.7), where the raising operator raises the eigenvalue states from $-\bar{h}$ through 0 to $\bar{h}$, and vice-versa for the lowering operators. In fact, if the eigenvalue C is equated with Planck's constant, i.e. $C = \bar{h}$, then the commutation relations between $\{\mathbf{A}_+, \mathbf{A}_0, \mathbf{A}_-\}$ are found to be identical to those of the angular momentum matrices $\mathbf{J}_x$, $\mathbf{J}_+$ and $\mathbf{J}_-$ in QM [5] (or $\mathbf{L}_x$ in [2]), i.e.

(2.12)
$$[\mathbf{J}_x, \mathbf{J}_+] = \bar{h}\mathbf{J}_+ \sim [\mathbf{A}_0, \mathbf{A}_+] = C\mathbf{A}_+$$
$$[\mathbf{J}_x, \mathbf{J}_-] = -\bar{h}\mathbf{J}_- \sim [\mathbf{A}_0, \mathbf{A}_-] = -C\mathbf{A}_-$$
$$[\mathbf{J}_+, \mathbf{J}_-] = 2\bar{h}\mathbf{J}_x \sim [\mathbf{A}_+, \mathbf{A}_-] = 2C\mathbf{A}_0\,.$$

Thus, the three URMT **A** matrices correspond to the following angular momentum matrices:

(2.13)
$\mathbf{A}_0 \sim \mathbf{J}_x$ a generator of rotations about the x-axis
$\mathbf{A}_+ \sim \mathbf{J}_+$ raising operator
$\mathbf{A}_- \sim \mathbf{J}_-$ lowering operator
$C \sim \bar{h}$ eigenvalue

Given that QM angular momentum and spin are mathematically the same concepts and that, equally, the same as the more abstract notion of nucleon isospin [3], [4], then the link from URMT eigenvectors to quark state vectors is easily made.

Quarks and Isospin

Using the following symbolic definitions:

u up quark state vector

d down quark state vector

$\mathbf{s}$ strange quark state vector

$\mathbf{I}_3$ operator for the component of isospin along a chosen axis

I_3 eigenvalue for the component of isospin along a chosen axis

then the isospin eigenvector equations [3] are

(2.14)
$$\mathbf{I}_3\mathbf{u} = \frac{1}{2}\mathbf{u},\ I_3 = \frac{1}{2}$$
$$\mathbf{I}_3\mathbf{s} = 0,\ I_3 = 0$$
$$\mathbf{I}_3\mathbf{d} = -\frac{1}{2}\mathbf{d},\ I_3 = -\frac{1}{2}.$$

Thus, the up quark has a component of isospin of ½, the down quark -½, and the strange quark zero.

Note that using a subscript '3' on $\mathbf{I}_3$ and I_3 implies the 'chosen' axis is the third, *z*-axis. However, in this paper, but not [1], it is actually the *x*-axis due to the attribution of $\mathbf{A}_0$ with the *x*-axis (2.13), and consistent with spin in [1]. As a consequence, $\mathbf{I}_3$ and I_3 might be better subscripted with a '1' as in $\mathbf{I}_1$ and I_1. The discrepancy is syntactic only, and fully resolved in [1].

By defining these three quarks in terms of the URMT eigenvectors as follows:

$$(2.15)\ \mathbf{u} = \frac{\mathbf{X}_+}{\sqrt{2}C},\ \mathbf{s} = \frac{\mathbf{X}_0}{C},\ \mathbf{d} = \frac{\mathbf{X}_-}{\sqrt{2}C},$$

and the operator $\mathbf{I}_3$ in terms of the URM3 equivalent operator $\mathbf{A}_0$, i.e.

$$(2.16)\ \mathbf{I}_3 = \frac{\mathbf{A}_0}{2C},$$

then the URM3 eigenvector equations (2.7), reproduced below, are identical to the isospin equations above, i.e.

(2.17)
$$\mathbf{I}_3\mathbf{u} = \frac{1}{2}\mathbf{u} \ \equiv\ \mathbf{A}_0\mathbf{X}_+ = C\mathbf{X}_+$$
$$\mathbf{I}_3\mathbf{s} = 0 \ \equiv\ \mathbf{A}_0\mathbf{X}_0 = 0$$
$$\mathbf{I}_3\mathbf{d} = -\frac{1}{2}\mathbf{d} \ \equiv\ \mathbf{A}_0\mathbf{X}_- = -C\mathbf{X}_-.$$

Note that the $\sqrt{2}$ factor on the quark eigenvectors is a normalisation constant, but the actual eigenvector elements remain integer only. For a numeric example, see Section (7).

As an example of (2.17), using the above definitions of $\mathbf{I}_3$ and **u**, the product $\mathbf{I}_3\mathbf{u}$ is

(2.18) $\mathbf{I}_3\mathbf{u} = \dfrac{\mathbf{A}_0}{2C}\dfrac{\mathbf{X}_+}{\sqrt{2}C}$,

and using the above eigenvector equation $\mathbf{A}_0\mathbf{X}_+ = C\mathbf{X}_+$ (2.7), and definition of $\mathbf{X}_+$ in terms of **u** (2.15), then the standard isospin eigenvector equation $\mathbf{I}_3\mathbf{u} = \mathbf{u}/2$ (2.14) is obtained.

As regards the two additional raising and lowering isospin operators $\mathbf{I}_+$ and $\mathbf{I}_-$, these operate on the quarks as follows:

(2.19)

$\mathbf{I}_+\mathbf{u} = 0$, $\mathbf{I}_+\mathbf{s} = 0$, $\mathbf{I}_+\mathbf{d} = \mathbf{u}$

$\mathbf{I}_-\mathbf{u} = \mathbf{d}$, $\mathbf{I}_-\mathbf{s} = 0$, $\mathbf{I}_-\mathbf{d} = 0$.

It is seen from these relations that $\mathbf{I}_+$ raises a down quark to an up quark, and annihilates the up quark, whereas $\mathbf{I}_-$ lowers an up quark to a down quark and annihilates the down quark; any other quark, such as the strange quark here, is always annihilated. Therefore, these standard QM isospin operators are effectively two-state, i.e.

(2.20) $\mathbf{I}_+ : \mathbf{d} \to \mathbf{u} \to 0$, $\mathbf{I}_- : \mathbf{u} \to \mathbf{d} \to 0$, $\mathbf{I}_+^2 = 0$, $\mathbf{I}_-^2 = 0$,

and, as a consequence, there is not a direct correspondence between $\mathbf{I}_+$ and $\mathbf{I}_-$, and URMT's three-state operators $\mathbf{A}_+$ and $\mathbf{A}_-$ (2.11). In fact, the direct correspondence comes when studying the lesser URMT theory that is URM2, i.e. 2x2 matrices (~ Pauli spin matrices [2]), and is fully detailed in [1] – note SU(2) ~ URM2 and SU(3) ~ URM3.

Nevertheless, URM3-equivalent operators of $\mathbf{I}_+$ and $\mathbf{I}_-$ can, and are, (see (2.34) further below) easily formed from the URMT eigenvectors, but first require the introduction of anti-particles into URMT.

Anti-particles and URMT Conjugate (row) Eigenvectors

Antiparticles are easily accommodated in URMT using the conjugate/reciprocal row-eigenvector equivalents of the standard, column vector forms $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$.

By defining a '**T** operator' matrix in URM3 as:

(2.21) $\mathbf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, $\mathbf{T} = \mathbf{T}^T = \mathbf{T}^{-1}$, $\mathbf{T}^2 = \mathbf{I}$,

then the reciprocal, row eigenvectors $\mathbf{X}^-$, $\mathbf{X}^0$ and $\mathbf{X}^+$ are defined in terms of the column eigenvector forms by:

(2.22) $\mathbf{X}^- = (\mathbf{T}\mathbf{X}_+)^T$, $\mathbf{X}^0 = (\mathbf{T}\mathbf{X}_0)^T$, $\mathbf{X}^+ = (\mathbf{T}\mathbf{X}_-)^T$.

These row eigenvectors $\mathbf{X}^-$, $\mathbf{X}^0$ and $\mathbf{X}^+$ are also known as the conjugates of $\mathbf{X}_+$, $\mathbf{X}_0$ and $\mathbf{X}_-$ respectively, where the conjugate of the minus eigenvector $\mathbf{X}_-$ is the plus row eigenvector $\mathbf{X}^+$, and vice versa.

Note that the **T** operator is the same as the Minkowski '2+1' metric in Special Relativity, to within sign convention, and is responsible for the Pythagorean nature of URMT when under Pythagoras conditions [7].

Using **T**, and eigenvector definitions (2.2), the conjugate eigenvectors are thus

(2.23) $\mathbf{X}^- = \begin{pmatrix} x & y & -z \end{pmatrix}$, $\mathbf{X}^0 = \begin{pmatrix} P & -Q & -R \end{pmatrix}$, $\mathbf{X}^+ = \begin{pmatrix} \alpha & \beta & \gamma \end{pmatrix}$.

These conjugate eigenvectors satisfy the following eigenvector relations, which will come in useful later when looking at anti-quarks:

(2.24)

$\mathbf{X}^+\mathbf{A}_+ = -2C\mathbf{X}^0$	$\mathbf{X}^0\mathbf{A}_+ = C\mathbf{X}^-$	$\mathbf{X}^-\mathbf{A}_+ = 0$
$\mathbf{X}^+\mathbf{A}_0 = C\mathbf{X}^+$	$\mathbf{X}^0\mathbf{A}_0 = 0$	$\mathbf{X}^-\mathbf{A}_0 = -C\mathbf{X}^-$
$\mathbf{X}^+\mathbf{A}_- = 0$	$\mathbf{X}^0\mathbf{A}_- = -C\mathbf{X}^+$	$\mathbf{X}^-\mathbf{A}_- = 2C\mathbf{X}^0$

Inner (or dot) Products

The inner (or dot) product of two vectors, giving a scalar result, is written throughout this paper as the product of a conjugate or reciprocal row-vector and a column vector, without the explicit 'dot' notation. For example, the inner product of the row vector $\mathbf{X}^+$ and column vector $\mathbf{X}_-$ is written as $\mathbf{X}^+\mathbf{X}_-$ instead of the more usual $\mathbf{X}^+ \cdot \mathbf{X}_-$, i.e.

(2.25) $\mathbf{X}^+\mathbf{X}_- = \mathbf{X}^+ \cdot \mathbf{X}_-$.

With this notation in mind, the inner products amongst the eigenvectors $\{\mathbf{X}^+, \mathbf{X}^0, \mathbf{X}^-\}$ and $\{\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-\}$ give the following six conservation equations, which are essential for most URMT quark algebra throughout:

(2.26)

$\mathbf{X}^-\mathbf{X}_+ = x^2 + y^2 - z^2 = 0$, Pythagoras equation
$\mathbf{X}^+\mathbf{X}_- = \alpha^2 + \beta^2 - \gamma^2 = 0$, Pythagoras equation
$\mathbf{X}^0\mathbf{X}_0 = P^2 + Q^2 - R^2 = C^2$, the DCE (2.4)
$\mathbf{X}^+\mathbf{X}_+ = \mathbf{X}^-\mathbf{X}_- = \alpha x + \beta y + \gamma z = 2C^2$, Potential equation for $V = 0$ [7]
$\mathbf{X}^0\mathbf{X}_+ = \mathbf{X}^-\mathbf{X}_0 = xP - yQ - zR = 0$, Delta equation
$\mathbf{X}^0\mathbf{X}_- = \mathbf{X}^+\mathbf{X}_0 = \alpha P - \beta Q + \gamma R = 0$, Dual delta equation

The reasoning behind the naming of these equations is briefly as follows: evidently the first two satisfy the Pythagoras equation; the third 'DCE' equation is hyperbolic in the dynamical variables and derives directly from the Invariance Principle, Section (4); the fourth has a non-zero potential energy (when not under URM3 Pythagoras conditions); and the last two 'delta' equations also derive from their invariance to global delta variations in the dynamical variables [7].

Conjugate A Matrices

Unlike the URMT eigenvectors, the **A** matrices are self-conjugate (known as 'Hermitian-like' in URMT [7]), and so their conjugate forms are the same as their standard forms (2.1).

Anti-quarks

Using the following definitions

$\bar{\mathbf{u}}$ up anti-quark state vector

$\bar{\mathbf{d}}$ down anti-quark state vector

$\bar{\mathbf{s}}$ strange anti-quark state vector

$\bar{\mathbf{I}}_3$ anti-quark operator for the component of isospin along a chosen (x) axis

then the anti-quark, isospin eigenvector equations are

(2.27)

$$\bar{\mathbf{d}}\bar{\mathbf{I}}_3 = \frac{1}{2}\bar{\mathbf{d}},\ I_3 = \frac{1}{2},$$
$$\bar{\mathbf{s}}\bar{\mathbf{I}}_3 = 0,\ I_3 = 0,$$
$$\bar{\mathbf{u}}\bar{\mathbf{I}}_3 = -\frac{1}{2}\bar{\mathbf{u}},\ I_3 = -\frac{1}{2}.$$

The anti-quark operator for the component of isospin ($\bar{\mathbf{I}}_3$) is given by negating the sign of its standard particle form, i.e.

(2.28) $\bar{\mathbf{I}}_3 = -\mathbf{I}_3$.

The up, down and strange anti-quarks are defined in terms of the URM3 row-eigenvectors as

(2.29) $$\bar{\mathbf{u}} = \frac{\mathbf{X}^+}{\sqrt{2}C},\ \bar{\mathbf{s}} = \frac{\mathbf{X}^0}{C},\ \bar{\mathbf{d}} = \frac{\mathbf{X}^-}{\sqrt{2}C},$$

and using $\bar{\mathbf{I}}_3 = -\mathbf{I}_3$ (2.28) and definition (2.16), then the anti-quark isospin operator $\bar{\mathbf{I}}_3$ is given in terms of $\mathbf{A}_0$ by

(2.30) $$\bar{\mathbf{I}}_3 = -\frac{\mathbf{A}_0}{2C}.$$

Using the above URMT-equivalent definitions, then the URM3 eigenvector equations (2.24), reproduced below, are seen to be identical to the isospin equations (2.27), i.e.

(2.31)

$$\bar{\mathbf{d}}\mathbf{I}_3 = \frac{1}{2}\bar{\mathbf{d}} \equiv \mathbf{X}^-\mathbf{A}_0 = -C\mathbf{X}^-$$
$$\bar{\mathbf{s}}\mathbf{I}_3 = 0 \equiv \mathbf{X}^0\mathbf{A}_0 = 0$$
$$\bar{\mathbf{u}}\mathbf{I}_3 = -\frac{1}{2}\bar{\mathbf{u}} \equiv \mathbf{X}^+\mathbf{A}_0 = C\mathbf{X}^+ .$$

Inner Products

The following inner vector products between all quarks are obtained using the particle and anti-particle eigenvector forms, (2.15) and (2.29), and inner product forms (2.26). These products are used extensively to verify operator algebra herein.

Reminder. $\bar{\mathbf{u}} \sim \mathbf{X}^+$, $\mathbf{u} \sim \mathbf{X}_+$, $\mathbf{s} \sim \mathbf{X}_0$, $\bar{\mathbf{s}} \sim \mathbf{X}^0$, $\bar{\mathbf{d}} \sim \mathbf{X}^-$, $\mathbf{d} \sim \mathbf{X}_-$.

(2.32)

$$\bar{\mathbf{u}}\mathbf{u} = 1,\ \mathbf{X}^+\mathbf{X}_+ = 2C^2 \qquad \bar{\mathbf{u}}\mathbf{s} = 0,\ \mathbf{X}^+\mathbf{X}_0 = 0 \qquad \bar{\mathbf{u}}\mathbf{d} = 0,\ \mathbf{X}^+\mathbf{X}_- = 0$$
$$\bar{\mathbf{s}}\mathbf{u} = 0,\ \mathbf{X}^0\mathbf{X}_+ = 0 \qquad \bar{\mathbf{s}}\mathbf{s} = 1,\ \mathbf{X}^0\mathbf{X}_0 = C^2 \qquad \bar{\mathbf{s}}\mathbf{d} = 0,\ \mathbf{X}^0\mathbf{X}_- = 0$$
$$\bar{\mathbf{d}}\mathbf{u} = 0,\ \mathbf{X}^-\mathbf{X}_+ = 0 \qquad \bar{\mathbf{d}}\mathbf{s} = 0,\ \mathbf{X}^-\mathbf{X}_0 = 0 \qquad \bar{\mathbf{d}}\mathbf{d} = 1,\ \mathbf{X}^-\mathbf{X}_- = 2C^2$$

Outer Products

The outer product of two vectors, giving a matrix result, is written throughout this paper as the product of a column vector and a conjugate (or reciprocal) row-vector, without the explicit product symbol $\otimes$. For example, the outer product of the column vector $\mathbf{X}_-$ and row vector $\mathbf{X}^+$ is written as $\mathbf{X}_-\mathbf{X}^+$ instead of the more usual $\mathbf{X}_- \otimes \mathbf{X}^+$, i.e.

(2.33) $\mathbf{X}_-\mathbf{X}^+ = \mathbf{X}_- \otimes \mathbf{X}^+$.

All matrix operators given in this paper have an outer product construction, i.e. can be written as the linear sum of one or more outer products of the eigenvectors.

The Outer Product form of $\mathbf{I}_+$ and $\mathbf{I}_-$

Having obtained all particle and anti-particle eigenvector forms, the raising and lowering operators, $\mathbf{I}_+$ and $\mathbf{I}_-$, can now be given as outer products in terms of URMT eigenvectors, stated as follows:

(2.34)

$$\mathbf{I}_+ = \mathbf{u}\bar{\mathbf{d}},\ \mathbf{I}_+ = \frac{\mathbf{X}_+\mathbf{X}^-}{2C^2}$$
$$\mathbf{I}_- = \mathbf{d}\bar{\mathbf{u}}\ \ \mathbf{I}_- = \frac{\mathbf{X}_-\mathbf{X}^+}{2C^2}.$$

With these definitions, and using the inner products (2.32), the earlier raising and lowering operations on the quarks (2.19) can be verified. For example, using $\mathbf{X}^-\mathbf{X}_- = 2C^2$,

(2.35) $$\mathbf{I}_+\mathbf{d} = \frac{(\mathbf{X}_+\mathbf{X}^-)\mathbf{X}_-}{2\sqrt{2}C^3} = \frac{\mathbf{X}_+(\mathbf{X}^-\mathbf{X}_-)}{2\sqrt{2}C^3} = \frac{\mathbf{X}_+}{\sqrt{2}C} = \mathbf{u}.$$

The anti-particle raising and lowering operators are also related to their standard forms by negation as per $\bar{\mathbf{I}}_3 = -\mathbf{I}_3$ (2.28), i.e.

(2.36)

$$\bar{\mathbf{I}}_+ = -\mathbf{I}_+,\ \bar{\mathbf{I}}_+ = -\frac{\mathbf{X}_+\mathbf{X}^-}{2C^2}$$
$$\bar{\mathbf{I}}_- = -\mathbf{I}_-,\ \bar{\mathbf{I}}_- = -\frac{\mathbf{X}_-\mathbf{X}^+}{2C^2}.$$

Using these and the inner products (2.32), the isospin raising, lowering operator actions on the anti-quarks are as given in the textbooks [3], i.e.

(2.37)

$$\bar{\mathbf{u}}\bar{\mathbf{I}}_+ = -\bar{\mathbf{d}},\ \bar{\mathbf{s}}\bar{\mathbf{I}}_+ = 0,\ \bar{\mathbf{d}}\bar{\mathbf{I}}_+ = 0$$
$$\bar{\mathbf{u}}\bar{\mathbf{I}}_- = 0,\ \bar{\mathbf{s}}\bar{\mathbf{I}}_- = 0,\ \bar{\mathbf{d}}\bar{\mathbf{I}}_- = -\bar{\mathbf{u}}.$$

For example, the last equation $\bar{\mathbf{d}}\bar{\mathbf{I}}_-$ evaluates as follows, in terms of URMT eigenvectors:

(2.38) $$\bar{\mathbf{d}}\bar{\mathbf{I}}_- = -\frac{\mathbf{X}^-(\mathbf{X}_-\mathbf{X}^+)}{2\sqrt{2}C^3} = -\frac{(\mathbf{X}^-\mathbf{X}_-)\mathbf{X}^+}{2\sqrt{2}C^3} = -\frac{\mathbf{X}^+}{\sqrt{2}C} = -\bar{\mathbf{u}}.$$

The Eigenvector form of $\mathbf{I}_3$ and $\mathbf{A}_0$

The $\mathbf{I}_3$ operator is given in terms of its eigenvectors, as outer products, by

(2.39) $$\mathbf{I}_3 = \frac{1}{2}(\mathbf{u}\bar{\mathbf{u}} - \mathbf{d}\bar{\mathbf{d}}),$$

which is actually a spectral decomposition [13], and using the definitions (2.15) and (2.29) the URM3-equivalent operator to $\mathbf{I}_3$, i.e. $\mathbf{A}_0$ (2.16), is written in terms of its eigenvectors as follows:

(2.40) $\mathbf{A}_0 = \frac{1}{2C}\left(\mathbf{X}_+\mathbf{X}^+ - \mathbf{X}_-\mathbf{X}^-\right)$ c.f. $\mathbf{I}_3 = \frac{\mathbf{A}_0}{2C}$ (2.16).

These two forms of $\mathbf{I}_3$ and $\mathbf{A}_0$, used in conjunction with the inner products (2.32), can be used to verify the isospin eigenvectors equations (2.17) and (2.31). In particular, because $\mathbf{I}_3$ has no strange or strange anti-quark dependence, then $\mathbf{I}_3\mathbf{s} = 0$ (2.17) and $\bar{\mathbf{s}}\bar{\mathbf{I}}_3 = 0$ (2.31) are verified. Similarly, in URMT, $\mathbf{A}_0$ has no $\mathbf{X}_0$ or $\mathbf{X}^0$ dependence, and so $\mathbf{A}_0\mathbf{X}_0 = 0$ (2.17) and $\mathbf{X}^0\mathbf{A}_0 = 0$ (2.31).

Note that, for later reference, the plus and minus matrices also have the following eigenvector constructions:

(2.41) $\mathbf{A}_+ = \frac{1}{C}\left(\mathbf{X}_0\mathbf{X}^- - \mathbf{X}_+\mathbf{X}^0\right)$, $\mathbf{A}_- = \frac{1}{C}\left(\mathbf{X}_-\mathbf{X}^0 - \mathbf{X}_0\mathbf{X}^+\right)$

Summary so far

Thus far, the three quarks up, down and strange, and their anti-particles, have been given URMT eigenvector representations, together with the three isospin operators $\mathbf{I}_3$, $\mathbf{I}_+$ and $\mathbf{I}_-$. As a consequence, all isospin operator actions on the quarks, such as obtaining their isospin component I_3, and raising and lowering their states, have also been given a URMT-equivalent.

3 A three-fold Solution

Given that three quarks can easily be represented within URM3, it might seem enough to now move on to six quarks. However, the earlier comparison of URMT operators with angular momentum shows that the URMT Pythagoras solution is incomplete, i.e. only the single, unity root matrix $\mathbf{A}_0$ has been attributed to a generator of rotations about a single axis (the *x* axis (2.13)), whilst the full treatment of angular momentum in both classical and quantum mechanics naturally considers all three Cartesian axes *x*, *y* and *z*. The result of this is that URM3 can be expanded, by analogy with angular momentum, to a three-axis solution, and with it comes the concept of quark colour, i.e. red, green and blue (RGB). Note that what will be seen to be a three-fold, RGB degeneracy in the URM3 solution is not the same as that of three quark generations, which also comes in URMT when later extending to six quarks, i.e. URMT embraces both quark colour and three-fold degeneracy.

Two new Matrices and Eigenvectors

Returning to the definitions of the QM raising and lowering operators of angular momentum, $\mathbf{J}_+$ and $\mathbf{J}_-$ (c.f. $\mathbf{I}_+$ and $\mathbf{I}_-$ for isospin), they are defined in terms of the generators $\mathbf{J}_y$ and $\mathbf{J}_z$ for rotations about the *y* and *z* axis [5] by

(3.1) $\mathbf{J}_+ = \mathbf{J}_y + i\mathbf{J}_z$, $\mathbf{J}_- = \mathbf{J}_y - i\mathbf{J}_z$.

However, no equivalent URM3 **A** matrices have been given for $\mathbf{J}_y$ and $\mathbf{J}_z$, only the founding unity root matrix $\mathbf{A}_0$, which has already been equated to the generator ($\mathbf{J}_x$) of rotations about the *x*-axis (2.13). Nevertheless, the raising and lowering operators $\mathbf{J}_+$ and $\mathbf{J}_-$ have also been equivalenced to URMT's matrices $\mathbf{A}_+$ and $\mathbf{A}_-$ (2.13), so that simply by replacing $\mathbf{J}_+$ and $\mathbf{J}_-$ with $\mathbf{A}_+$ and $\mathbf{A}_-$, and rearranging, two new **A** matrices, labelled $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$, are obtained as defined by

(3.2)

$$\mathbf{A}_{D0} = \frac{i}{2}\left(\mathbf{A}_+ - \mathbf{A}_-\right), \sim \mathbf{J}_y, \text{ y-axis}$$

$$\mathbf{A}_{S0} = \frac{1}{2}\left(\mathbf{A}_+ + \mathbf{A}_-\right), \sim \mathbf{J}_z, z \text{ axis.}$$

Note that, ordinarily, these new **A** matrices $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$ might be subscripted with *y* and *z* exactly as per $\mathbf{J}_y$ and $\mathbf{J}_z$ but, for various reasons, this is not possible without conflicting or causing confusion with other URMT notation. The subscripts '$_D$' and '$_S$' actually denote difference and sum respectively, and the trailing subscript zero identifies that these two matrices have similar properties to $\mathbf{A}_0$, not least because $\mathbf{A}_0$, $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$ are the URMT equivalents of the generator matrices $\mathbf{J}_x$, $\mathbf{J}_y$ and $\mathbf{J}_z$, as in

(3.3) $\mathbf{A}_0 \sim x$-axis, $\mathbf{A}_{D0} \sim y$-axis, $\mathbf{A}_{S0} \sim z$-axis.

Because of the common subscript of '0' on all three matrices, $\mathbf{A}_0$, $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$, they are known as 'zero' matrices in URMT – but this is not because they are trivially zero.

Just like the founding unity root matrix $\mathbf{A}_0$, there exists zero eigenvectors (i.e. with a zero eigenvalue) for $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$, denoted by $\mathbf{X}_{D0}$ and $\mathbf{X}_{S0}$ respectively, satisfying their defining equations

(3.4) $\mathbf{A}_{D0}\mathbf{X}_{D0} = 0$, $\mathbf{A}_{S0}\mathbf{X}_{S0} = 0$,

and constructed in a similar way to their associated matrices (3.2), i.e.

(3.5) $\mathbf{X}_{D0} = \frac{i}{2}\left(\mathbf{X}_+ - \mathbf{X}_-\right)$, $\mathbf{X}_{S0} = \frac{1}{2}\left(\mathbf{X}_+ + \mathbf{X}_-\right)$.

The reciprocals (or conjugates) of $\mathbf{X}_{D0}$ and $\mathbf{X}_{S0}$ are defined in the standard way (2.22) as

(3.6) $\mathbf{X}^{D0} = (\mathbf{T}\mathbf{X}_{D0})^T$, $\mathbf{X}^{S0} = (\mathbf{T}\mathbf{X}_{S0})^T$.

From the above definition of $\mathbf{X}_{D0}$ and $\mathbf{X}_{S0}$, the URMT conjugates $\mathbf{X}^{D0}$ and $\mathbf{X}^{S0}$ are thus given by

(3.7) $\mathbf{X}^{D0} = \frac{i}{2}\left(\mathbf{X}^{-} - \mathbf{X}^{+}\right),\ \mathbf{X}^{S0} = \frac{1}{2}\left(\mathbf{X}^{-} + \mathbf{X}^{+}\right).$

With these definitions of $\mathbf{X}^{D0}$, $\mathbf{X}^{S0}$ and $\mathbf{X}_{D0}$, $\mathbf{X}_{S0}$, and using the inner products (2.26), the following additional eigenvector inner product relations can be determined

(3.8)

$\mathbf{X}^{D0}\mathbf{X}_{D0} = C^2,\ \mathbf{X}^{S0}\mathbf{X}_{S0} = C^2$ hyperbolic as for $\mathbf{X}_0$ (2.26)

$\mathbf{X}^{D0}\mathbf{X}_{S0} = 0,\ \mathbf{X}^{S0}\mathbf{X}_{D0} = 0$, orthogonal.

Commutation Relations

The commutations relations between $\mathbf{A}_0$, $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$ are stated as follows, with the corresponding, equivalent angular momentum relations for the **J** matrices given on the right:

(3.9)

$$\left[\mathbf{A}_0, \mathbf{A}_{D0}\right] = iC\mathbf{A}_{S0} \sim \left[\mathbf{J}_x, \mathbf{J}_y\right] = i\bar{h}\mathbf{J}_z$$
$$\left[\mathbf{A}_{D0}, \mathbf{A}_{S0}\right] = iC\mathbf{A}_0 \sim \left[\mathbf{J}_y, \mathbf{J}_z\right] = i\bar{h}\mathbf{J}_x$$
$$\left[\mathbf{A}_{S0}, \mathbf{A}_0\right] = iC\mathbf{A}_{D0} \sim \left[\mathbf{J}_z, \mathbf{J}_x\right] = i\bar{h}\mathbf{J}_y.$$

These relations can be verified using the definitions of $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$ (3.2), in terms of $\mathbf{A}_+$ and $\mathbf{A}_-$, and then using the existing commutation relations (2.12). Evidently, from these relations, if C and $\bar{h}$ are equated, the correspondence between the **A** and **J** matrices is exact.

Three sets of Matrices and Eigenvectors

Thus far, URM3 has now been extended to three zero, **A** matrices, $\mathbf{A}_0$, $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$. In fact, $\mathbf{A}_{D0}$ and $\mathbf{A}_{S0}$ have identical eigenvalues to $\mathbf{A}_0$, i.e. C,0,$-C$ (2.7) and come with their own plus and minus eigenvector equivalents of $\mathbf{X}_+$ and $\mathbf{X}_-$ (for $\mathbf{A}_0$), given in terms of the base set of eigenvectors $\{\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-\}$, defined as follows:

(3.10)

$$\mathbf{X}_{D+} = \frac{1}{2}\left(\mathbf{X}_+ + \mathbf{X}_-\right) + i\mathbf{X}_0,\ \mathbf{X}_{D-} = \frac{1}{2}\left(\mathbf{X}_+ + \mathbf{X}_-\right) - i\mathbf{X}_0,\ \mathbf{X}_{D-} = \mathbf{X}_{D+}^{*}$$
$$\mathbf{X}_{S+} = \frac{1}{2}\left(\mathbf{X}_+ - \mathbf{X}_-\right) - \mathbf{X}_0,\ \mathbf{X}_{S-} = -\frac{1}{2}\left(\mathbf{X}_+ - \mathbf{X}_-\right) - \mathbf{X}_0$$

Accordingly, these eigenvectors can also be associated with new matrices $\mathbf{A}_{D+}$, $\mathbf{A}_{D-}$, $\mathbf{A}_{S+}$ and $\mathbf{A}_{S-}$, defined in terms of the base set $\{\mathbf{A}_0, \mathbf{A}_+, \mathbf{A}_-\}$ in an identical way to the above eigenvectors, i.e.

(3.11)

$$\mathbf{A}_{D+} = \frac{1}{2}(\mathbf{A}_+ + \mathbf{A}_-) + i\mathbf{A}_0,\ \mathbf{A}_{D-} = \frac{1}{2}(\mathbf{A}_+ + \mathbf{A}_-) - i\mathbf{A}_0,\ \ \mathbf{A}_{D-} = \mathbf{A}_{D+}^*$$

$$\mathbf{A}_{S+} = \frac{1}{2}(\mathbf{A}_+ - \mathbf{A}_-) - \mathbf{A}_0,\ \mathbf{A}_{S-} = -\frac{1}{2}(\mathbf{A}_+ - \mathbf{A}_-) - \mathbf{A}_0.$$

In fact, these new matrices have eigenvector relations that are identical to those, (2.7) to (2.10), for the base set, i.e., to get the difference and sum set relations, just add the subscript '$_D$' or '$_S$' respectively, so that $\mathbf{A}_0$ becomes $\mathbf{A}_{D0}$ or $\mathbf{A}_{S0}$, $\mathbf{A}_+$ becomes $\mathbf{A}_{D+}$ or $\mathbf{A}_{S+}$, and $\mathbf{A}_-$ becomes $\mathbf{A}_{D-}$ or $\mathbf{A}_{S-}$. Exactly the same subscript replacements apply to the eigenvectors, e.g. $\mathbf{X}_+$ becomes $\mathbf{X}_{D+}$ or $\mathbf{X}_{S+}$ etc. Not only are all eigenvector relations the same for base, difference and sum sets, but so too are the inner products as per (2.26) and commutation relations (2.12). Lastly, the construction of the difference and sum **A** matrices, in terms of the eigenvectors, is also the same as those for the base set (2.40) and (2.41). See [1] for full details of all relations and definitions for the difference and sum sets.

Summarising...

So, by analogy with angular momentum, URMT has been expanded from a single axis solution in the base set, to a three-axis solution comprising the base, difference and sum sets.

base matrices $\{\mathbf{A}_0, \mathbf{A}_+, \mathbf{A}_-\}$, eigenvectors $\{\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-\}$, *x*-axis

difference matrices $\{\mathbf{A}_{D0}, \mathbf{A}_{D+}, \mathbf{A}_{D-}\}$, eigenvectors $\{\mathbf{X}_{D+}, \mathbf{X}_{D0}, \mathbf{X}_{D-}\}$, *y*-axis

sum matrices $\{\mathbf{A}_{S0}, \mathbf{A}_{S+}, \mathbf{A}_{S-}\}$, eigenvectors $\{\mathbf{X}_{S+}, \mathbf{X}_{S0}, \mathbf{X}_{S-}\}$, *z*-axis

Each set is algebraically identical to each other, e.g. swapping the subscript say, '$_{D0}$' with '$_{S0}$', will give the same equations, i.e. there is no preferred or special axis, which means that any set can be used to represent the three quarks (so far), i.e. up, down and strange, with no preferred set, and neither does this property disappear when expanding to all six-quarks.

Only one of the sets is linearly independent, and the difference and sum sets are given further above, in terms of the base set (*x*-axis) but, for example, the base and difference sets could equally well be written in terms of the sum set. Whilst the sets may not be linearly independent, the actual eigenvector and matrices are numerically distinct from each other, e.g. $\mathbf{A}_0$ and $\mathbf{A}_{D0}$ are not numerically the same matrices, which means that the differing quark colours are distinct.

Because any one of the three sets could equally represent the up, down and strange, there is a three-fold redundancy, which is attributed here to each quark coming in one of three colours, red, green and blue, as per the theory of Quantum Chromodynamics (QCD). This RGB expansion of URMT is not the same as the three-fold redundancy in quark families, (up, down), (charm, strange) and (bottom, top), which is embraced in URMT when expanding to six quarks, i.e. URM6, Section (5).

4 Eigenvector Evolution and Unitary Transforms

This section shows that URMT's parametric, eigenvector evolution takes the same, exponentiated form as a unitary transform in QM, and therefore retains all inner products. A comparison with the time-dependent evolution of the wavefunction shows that the evolution, which is ultimately attributable to URMT's Invariance Principle (further below), can be interpreted as an action principle.

The URM3 Parametric Solution

URM3 is a completely solved problem with a parametric solution for all eigenvectors. For example, the eigenvector $\mathbf{X}_+$, comprising the triplet (x, y, z), is given in terms of arbitrary integers k and l by the standard parameterization of a Pythagorean triple, i.e.

(4.1) $x = 2kl,\ y = (l^2 - k^2),\ z = (l^2 + k^2), k, l \in \mathbb{Z}.$

The $\mathbf{X}_0$ and $\mathbf{X}_-$ eigenvectors are also solved in terms of both k, l and an additional time parameter *t*; see [7] or [10] for more details. Ultimately this solution means that $\mathbf{X}_0$ and $\mathbf{X}_-$ evolve with time *t*, whilst $\mathbf{X}_+$ remains static, as in the following eigenvector evolution equations, where the superscript prime denotes an initial value at time $t = 0$:

(4.2)
$\mathbf{X}_+ = \mathbf{X}'_+$, static - no *t* dependence
$\mathbf{X}_0 = -t\mathbf{X}'_+ + \mathbf{X}'_0$
$\mathbf{X}_- = -t^2\mathbf{X}'_+ + 2t\mathbf{X}'_0 + \mathbf{X}'_-.$

Note here that the eigenvectors are related to each other via calculus relations, i.e. $\mathbf{X}_+$ is the negative of the time derivative of $\mathbf{X}_0$, and $\mathbf{X}_0$ is twice the time derivative of $\mathbf{X}_-$. Whilst this fact is not used in this paper, it is of general importance in URMT physics.

Clearly, since the three eigenvectors have been put in a one-to-one correspondence with the quarks (2.15), so too can the quarks state vectors therefore evolve, and it could be that at some particular time (likely very small) in the universe's evolution the quarks froze (settled) at a specific numeric value, i.e. that which we see today. Note that the association of parameter *t* with time is a physical interpretation, it could be an arbitrary integer with no physical significance, but assigning it to a temporal parameter fits very well within the Standard Physical Interpretation (SPI) of URMT [7], and the aforementioned calculus relations amongst the eigenvectors. Ultimately, this parametric freedom (in *k*,*l* and *t*) leads to the ability to tune the quark representation such that its eigenvectors may then actually have physically significant properties such as mass, and not just be an arbitrary algebraic representation with no obvious physical significance. To spell this out, whilst the eigenvector solution set can take an infinite set of values, there may be one specific set of parameters *k*,*l*,*t*, which nature has somehow picked-out to give the quarks their properties.

There is also a frequency domain equivalent of the above time-domain evolution, with free parameter *f*. This frequency domain evolution evolves the $\mathbf{X}_+$ and $\mathbf{X}_0$ eigenvectors, leaving

the $\mathbf{X}_-$ eigenvector static, i.e. frequency independent, and this is a true dual formulation of URM3, where $\mathbf{X}_-$ is considered the dual of $\mathbf{X}_+$ and vice-versa [7].

The base set of initial eigenvectors $\{\mathbf{X}'_+, \mathbf{X}'_0, \mathbf{X}'_-\}$ can be taken from any particular solution in *k*, *l* and *t*, and a full quark representation given in terms of them. In other words, the quark representation is not unique and depends upon both the initial solution (*k*,*l* parameterisation), and the time *t*.

Lastly since the difference and sum sets are dependent upon the base set, they too evolve. Of course, the evolution could also start with the difference or sum set instead. It is merely URM3 convention that the base set is used since it is historically the first and only solution that existed right up until this work on quarks.

Because the eigenvectors evolve, so too do the **A** matrices by their eigenvector constructions (2.40) and (2.41), and this leads to the same form of evolution equations for these matrices as per the eigenvectors, i.e.

(4.3)
$\mathbf{A}_+ = \mathbf{A}'_+$ static - no *t* dependence.
$\mathbf{A}_0 = -t\mathbf{A}_+ + \mathbf{A}'_0$, where $\mathbf{A}_0 = \mathbf{A}'_0$ at $t = 0$.
$\mathbf{A}_- = -t^2\mathbf{A}_+ + 2t\mathbf{A}'_0 + \mathbf{A}'_-$.

The Invariance Principle

Of particular note in the above is the evolution of the equation for $\mathbf{A}_0$, which is really a statement of URMT's founding Invariance Principle - an explanation follows:

The dynamical equations and their solutions are invariant
to a coordinate transformation in the dynamical variables.

The *dynamical equations* are just those three represented by the founding, URMT 'invariant' eigenvector equation $\mathbf{A}_0\mathbf{X}_+ = C\mathbf{X}_+$ (2.7), and the *coordinate transformation in the dynamical variables* (P, Q, R) is given by expanding $\mathbf{A}_0 = -t\mathbf{A}_+ + \mathbf{A}'_0$ (4.3) into the elements P, Q, R of $\mathbf{A}_0$ and x, y, z of $\mathbf{A}_+$ to give

(4.4) $\mathbf{A}_0 = -t\mathbf{A}_+ + \mathbf{A}'_0 \sim$
$P = P' - tx$, $Q = Q' + ty$, $R = R' - tz$, where $P', Q', R' = P, Q, R$ at $t = 0$.

It can be seen that the dynamical variables P, Q, R retain their congruence definition (2.5) under this transformation, which is the number-theoretic basis upon which URMT is founded. Indeed, URMT can be derived with this principle as the starting point. A key reason to mention this point, which may seem out of place, is that URMT's time-domain evolution is really just a statement of the congruential nature of the dynamical variables as unity roots – it is pure number theory, no more no less. Yet, as will be seen next, this evolution can be expressed in an exponentiated, unitary form common to SU(*n*) and QM. Before commencing, note too that, once again, there is a dual, frequency domain form of the Invariance Principle

that is also due to the invariance of the dynamical variable to variations in a frequency parameter *f*, e.g. $P = P' - f\alpha$, $Q = Q' + f\beta$, $R = R' + f\gamma$, where α, β, γ are the elements of the static (frequency-independent) eigenvector $\mathbf{X}_-$ (2.2).

Exponential Evolution

The above three eigenvector evolution equations in $\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-$ (4.2) are the classic forms given in [7], but can now be more succinctly expressed in an exponentiated, unitary form as per QM time-domain evolution.

The eigenvector matrix $[\mathbf{X}]$ is defined as a row vector of the three, base column eigenvectors $\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-$, i.e.

(4.5) $[\mathbf{X}] = \begin{pmatrix} \mathbf{X}_+ & \mathbf{X}_0 & \mathbf{X}_- \end{pmatrix}$,

with the initial value at $t = 0$ denoted by a superscript prime, as in

(4.6) $[\mathbf{X}]' = \begin{pmatrix} \mathbf{X}'_+ & \mathbf{X}'_0 & \mathbf{X}'_- \end{pmatrix}$, $t = 0$,

An exponentiated evolution matrix $\mathbf{E}_t$ is defined in terms of URMT's plus matrix $\mathbf{A}_+$, time *t*, eigenvalue *C*, by

(4.7) $\mathbf{E}_t = \exp\left(t \frac{\mathbf{A}'_+}{C} \right)$,

where $\mathbf{A}_+ = \mathbf{A}'_+$, invariant by definition (4.3).

Using this evolution matrix $\mathbf{E}_t$, then the initial eigenvector matrix $[\mathbf{X}]'$ evolves according to the matrix product

(4.8) $[\mathbf{X}]_t = \mathbf{E}_t [\mathbf{X}]'$.

Note that there is also no pre-multiplying of the exponent in (4.7) by the unit imaginary number *i*, unlike a standard unitary transform, e.g. $e^{i\boldsymbol{\sigma}}$ in the case of a Pauli spin matrix $\boldsymbol{\sigma}$ [5].

Justification for (4.8) comes from first expanding the exponential (4.7) to give the following infinite sum of matrix terms

(4.9) $$\exp\left(t \frac{\mathbf{A}_+}{C} \right) = \lim_{n \to \infty} \mathbf{I} + \left(\frac{t\mathbf{A}_+}{C} \right) + \frac{1}{2!}\left(\frac{t\mathbf{A}_+}{C} \right)^2 + \frac{1}{3!}\left(\frac{t\mathbf{A}_+}{C} \right)^3 + \ldots \frac{1}{n!}\left(\frac{t\mathbf{A}_+}{C} \right)^n,$$

where $\mathbf{I}$ is the identity matrix.

The important point about the above series expansion is that matrix terms of order $\mathbf{A}_+^3 = 0$ and higher are all zero, i.e.

(4.10) $\mathbf{A}_+^n = 0,\ n \geq 3$.

This also applies in QM, but the standard SU(2) and SU(3) raising operator, e.g. $\mathbf{I}_+$ (2.34), actually squares (not cubes) to zero, i.e. $\mathbf{I}_+^2 = 0$ (2.20), noting that, in the case of URMT's raising operator $\mathbf{A}_+$, this has a non-zero square, i.e. $\mathbf{A}_+^2 \neq 0$, The same is true of the lowering operators, i.e. $\mathbf{I}_-^2 = 0$ (2.20), and $\mathbf{A}_-^3 = 0$, but $\mathbf{A}_-^2 \neq 0$. The squaring to zero is related to the two-state nature of QM spin, as for URM2 [1], but not URM3's three-state nature (2.11).

The outcome of this cubic 'nilpotency' is that the sum to infinity in (4.9) is reduced to just the three terms, and the evolution matrix $\mathbf{E}_t$ becomes

(4.11) $$\mathbf{E}_t = \mathbf{I} + \left(\frac{t\mathbf{A}_+}{C}\right) + \frac{1}{2}\left(\frac{t\mathbf{A}_+}{C}\right)^2.$$

In other words, what is potentially an infinite series expansion has a finite cut-off. Note that, for a standard unitary transformation [5], whilst the series expansion remains infinite, as too in URMT when using the 'zero' matrix generators $\{\mathbf{A}_0, \mathbf{A}_{S0}, \mathbf{A}_{D0}\}$, each element of the resulting matrix is a trigonometric series [12] – actually trigonometric in standard QM, but hyperbolic trig. i.e. cosh, sinh, in URMT.

As an example of using $\mathbf{E}_t$ (4.11), the evolution of initial eigenvector $\mathbf{X}'_-$ is given by

(4.12) $$\mathbf{E}_t\mathbf{X}'_- = \mathbf{X}'_- + \left(\frac{t}{C}\right)\mathbf{A}_+\mathbf{X}'_- + \frac{1}{2!}\left(\frac{t}{C}\right)^2\mathbf{A}_+\left(\mathbf{A}_+\mathbf{X}'_-\right),$$

and using $\mathbf{A}_+\mathbf{X}'_- = 2C\mathbf{X}'_0$ (2.10) this becomes

(4.13) $$\mathbf{E}_t\mathbf{X}'_- = \mathbf{X}'_- + 2t\mathbf{X}'_0 + \left(\frac{t^2}{C}\right)\mathbf{A}_+\mathbf{X}'_0.$$

Using $\mathbf{A}_+\mathbf{X}'_0 = -C\mathbf{X}'_+$ (2.10) gives

(4.14) $$\mathbf{E}_t\mathbf{X}'_- = \mathbf{X}'_- + 2t\mathbf{X}'_0 - t^2\mathbf{X}'_+,$$

and so $\mathbf{E}_t$ evolves the minus eigenvector $\mathbf{X}_-$ exactly as per the classic, algebraic form (4.2).

The evolution of the other two eigenvectors, $\mathbf{X}_+$ and $\mathbf{X}_0$, can be verified similarly [1].

The inverse of $\mathbf{E}_t$ is easily found since, using the relation,

(4.15) $$\exp(-\mathbf{A})\exp(\mathbf{A}) = e^{-\mathbf{A}}e^{\mathbf{A}} = \mathbf{I}.$$

then the inverse ($\mathbf{E}_t^{-1}$) of $\mathbf{E}_t$ is given by

(4.16) $\mathbf{E}_t^{-1} = \exp\left(-t\frac{\mathbf{A}'_+}{C}\right)$.

Note that (4.15) is actually only possible by virtue that the commutator of a matrix with itself is zero, i.e. $[\mathbf{A}, \mathbf{A}] = 0$, see the Baker-Campbell-Hausdorf identity [13].

By defining the conjugate evolution operator $\overline{\mathbf{E}}_t$ as follows:

(4.17) $\overline{\mathbf{E}}_t = \mathbf{T}\mathbf{E}_t{}^T\mathbf{T}^{-1}$, where $\mathbf{T}^{-1} = \mathbf{T} = \mathbf{T}^T$,

and the conjugate matrix $[\overline{\mathbf{X}}]$, comprising three rows, each a conjugate, row eigenvector i.e.

(4.18) $[\overline{\mathbf{X}}] = \begin{pmatrix} \mathbf{X}^- \\ \mathbf{X}^0 \\ \mathbf{X}^+ \end{pmatrix}$,

then the conjugate eigenvectors evolve according to

(4.19) $[\overline{\mathbf{X}}]_t = [\overline{\mathbf{X}}]'\overline{\mathbf{E}}_t$.

The conjugate operator $\overline{\mathbf{E}}_t$ is actually also the same as the inverse $\mathbf{E}_t^{-1}$ of $\mathbf{E}_t$, as in

(4.20) $\overline{\mathbf{E}}_t\mathbf{E}_t = \mathbf{I}$,

i.e. by (4.16),

(4.21) $\overline{\mathbf{E}}_t = \mathbf{E}_t^{-1} = \exp\left(-t\frac{\mathbf{A}_+}{C}\right)$.

The above equality of the conjugate $\overline{\mathbf{E}}_t$ and inverse $\mathbf{E}_t^{-1}$ shows that URMT conjugation is merely a sign change in the exponent, and not a full complex conjugation since there is no imaginary unit in the exponential. Most importantly though, URMT relates conjugate forms, i.e. the row eigenvectors, to anti-particles (2.29) and so complex conjugation has been replaced by a straightforward sign reversal, physically manifest above as a reversal in time, i.e. t to $-t$.

Thus, to summarise URMT eigenvector evolution, what was once three algebraic evolution equations (4.2), has now been re-written in an exponential form as per a standard unitary transform in QM, keeping in mind that the URMT evolution equations are merely just an expression of the Invariance Principle and, ultimately, due to the congruential nature of the unity roots. From a physical perspective the evolution of anti-particles is seen to be a time-reversal of their standard forms.

Time-domain Evolution and the Wave Function

With URMT evolution now written in an exponential, unitary form, it is intriguing to look at a very similar equation in QM.

For a time-independent Hamiltonian **H**, the wavefunction at time *t*, denoted by $\psi(t)$, is given in terms of its initial wavefunction, time $t = 0$, according to

$$(4.22)\quad \psi(t) = \exp(-it\frac{\mathbf{H}}{\bar{h}})\psi(0).$$

Given the wavefunction is a state vector, then comparing this with URMT's own time-domain eigenvector evolution (4.8), the following associations are made between QM and URMT

$$(4.23)\quad \psi(t) \sim [\mathbf{X}]_t,\ \psi(0) \sim [\mathbf{X}]',\ -i\frac{\mathbf{H}}{\bar{h}} \sim \frac{\mathbf{A}'_+}{C}.$$

The Hamiltonian **H** is time-independent, by definition, and so too the URMT raising operator $\mathbf{A}'_+$. The scaled Planck constant $\bar{h}$ is, of course, also constant by definition, and so too the URMT invariant eigenvalue *C*. The earlier association of $\bar{h}$ with *C* was made in connection with QM angular momentum (or spin) and URMT in (2.13), and now confirmed as above.

This comparison is quite remarkable given that, once again, URMT's time-domain evolution, is really a consequence of the Invariance Principle, which is pure number theory and just an esoteric statement that, due to the congruence relations (2.5), there is an infinite set of solutions parameterised by a numeric parameter, ascribed to time here for time-domain evolution.

The above comparison between URMT and QM leads to a modification of URMT's Standard Physical Interpretation (SPI) [7], known as the Quantum Physical Interpretation (QPI), where the units of *C* are now those of Planck's constant, and the units of $\mathbf{A}_+$ are those of the Hamiltonian i.e.

(4.24)
$units(\mathbf{A}_+) = units(\mathbf{H}) = J = L^2T^{-2}$, QPI energy
$units(C) = units(\bar{h}) = JT = L^2T^{-1}$, QPI action (see below)

with the ratios $\mathbf{A}_+ / C$ and $\mathbf{H}/\bar{h}$ possessing the units of frequency.

Note that this newer QPI is really just a slight adjustment to the more established SPI, which primarily speaks in terms energy-related quantities rather than action quantities, as now discussed.

A strong reason to use this new QPI is that the units of Planck's constant are those of the dynamical quantity known as action, and this leads to a subsequent reinterpretation of URMT's Invariance Principle in terms of action quantities.

Given the units of *C* are those of an action quantity, then so too are those of $\mathbf{A}_0$ since *C* and $\mathbf{A}_0$ both have the same units as the dynamical variables P,Q,R, as can be deduced from, say, (2.4). In addition, since the units of $\mathbf{A}_+$ are now those of energy (4.24), then the quantity $t\mathbf{A}_+$ is also that of an action quantity, and so the algebraic form of the Invariance Principle, i.e. $\mathbf{A}_0 = -t\mathbf{A}_+ + \mathbf{A}'_0$ (4.4), is thus a statement on an action quantity, albeit the principle is not per se a 'principle of least action', but more just a statement on the behaviour of the dynamical, eigenvector equations (2.7) under a change '$-t\mathbf{A}'_+$' in action, as above. Looking at the equivalent transformations in the dynamical variables P,Q,R (4.4), the principle states that the $\mathbf{X}_+$ eigenvector solution, comprising coordinates *x*,*y*,*z*, is invariant to a time-domain change in the action of the dynamical variables.

Thus, to summarise this section: URMT eigenvector evolution is unitary since all inner products between the time-evolved eigenvectors remain invariant to temporal variation; the anti-particle evolution of the conjugate eigenvectors evolves in a time-reversed manner; the evolution action on the eigenvectors is seen to be equivalent that of the time-domain evolution of the wavefunction; and the URMT invariance Principle that gives these aforementioned properties can be interpreted as an action principle.

5 The Six Quark Solution

This section expands the URMT three-quark scheme to all six, currently known quarks, plus their anti-quarks. The six quarks are usually given in pairs, i.e. up and down, strange and charm, bottom and top, each pair forming a 'generation', hence the concept of three generations of quarks. In nature, the first generation, comprising the up and down quarks, form the two stable nucleon particles, i.e. the proton and neutron, whilst the other two generations form relatively unstable particles with lifetimes of a maximum ten or so nanoseconds.

Thus, the up and down quarks hold a special position in our atomic world, and this too is reflected in the URMT model, whereby the up and down quarks are represented by eigenvectors that are Pythagorean in nature, whereas the other four quarks will all be seen to possess structurally different 'zero' eigenvectors, e.g. $\mathbf{X}_0$ (2.2), that are hyperbolic in nature, and with zero isospin, as opposed to non-zero isospin for the up and down quarks.

The zero-isospin, strange quark has already been equivalenced to the URM3 eigenvector $\mathbf{X}_0$ (2.15), leaving the three remaining, known quarks, charm, bottom and top to be given the same treatment. All four zero-isospin quarks are represented by more zero eigenvectors, which are obtained by recourse to the URMT method of 'lifting' solutions.

The URMT method of lifting solutions first appeared in [14], and is the process by which a higher order matrix solution, i.e. 6x6 here, can be obtained from a lower-order solution, that is 3x3 here. Specifically, the URM3 eigenvector solution for the up, down and strange quark is lifted to URM6, which preserves the original solution and adds another three additional zero eigenvectors (zero isospin eigenvalues) in the process, representing the charm, top and bottom quarks. Strictly speaking, it is not the original solution that is exactly preserved in its numeric form, but rather all the existing algebraic relations that remain invariant in the lifting process. This process adds more variational parameters, which can be used to tune the eigenvector

solution. The parametric tuning process is just an extension of URMT evolution, discussed in the previous section, but now also highlights the fact that all six quarks can be made to look like either the up or down quark, for large temporal parameters.

To keep the subject matter brief, only the time-domain, lifting method is presented, and readers are referred to [1] for the frequency-domain equivalent, which is almost identical, i.e. replace time *t* with frequency *f*, and swap $\mathbf{X}_+$ and $\mathbf{X}_-$.

The URM6 solution

Like its URM3 counterpart, this URM6 solution (below) is a completely solved problem, with an analytic solution and a full complement of six distinct eigenvectors, but only three unique eigenvalues, i.e. $\lambda = \pm C, 0$. The zero eigenvalue is repeated four times so that the complete set is $\lambda = \pm C, 0,0,0,0$. The up and down quarks are now represented by Pythagorean sextuplets, as opposed to triplets in the three-quark, URM3 scheme, and the remaining, four zero eigenvectors, representing the strange, charm, bottom and top quarks are hyperbolic sextuplets.

Starting with the initial URM3 solution, i.e. $\mathbf{X}'_{3+}$, $\mathbf{X}'_{30}$, $\mathbf{X}'_{3-}$ (4.2), where $\mathbf{X}_{3+}$ is invariant to time-domain variations in t_3 (formerly t (4.2)), i.e. $\mathbf{X}_{3+} = \mathbf{X}'_{3+}$, then the initial URM6 eigenvector solution is given by the following six eigenvectors (using block notation, where $\mathbf{0}_3$ is the three-element null vector):

(5.1)

$$\mathbf{X}'_{6+} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}'_{3+} \end{pmatrix}, \ \mathbf{X}'_{6-} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}'_{3-} \end{pmatrix}$$

$$\mathbf{X}'_{60A} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}'_{30} \end{pmatrix}, \ \mathbf{X}'_{60B} = \begin{pmatrix} 0 \\ 0 \\ C \\ \mathbf{0}_3 \end{pmatrix}, \ \mathbf{X}'_{60C} = \begin{pmatrix} 0 \\ C \\ 0 \\ \mathbf{0}_3 \end{pmatrix}, \ \mathbf{X}'_{60D} = \begin{pmatrix} C \\ 0 \\ 0 \\ \mathbf{0}_3 \end{pmatrix}.$$

These eigenvectors then evolve, in the time-domain according to the values of the four temporal parameters, t_3, t_4, t_5 and t_6, which are zero in the initial-value solution (and not therefore shown above)

(5.2)

$\mathbf{X}_{6+} = \mathbf{X}'_{6+}$, $\lambda = C$, static, no t_3, t_4, t_5, t_6 dependence

$$\mathbf{X}_{6-} = -(t_6^2 + t_5^2 + t_4^2 + t_3^2)\mathbf{X}_{6+} + 2t_3\mathbf{X}'_{60A} + 2t_4\mathbf{X}'_{60B} + 2t_5\mathbf{X}'_{60C} + 2t_6\mathbf{X}'_{60D} + \mathbf{X}'_{6-}, \ \lambda = -C$$

$$\mathbf{X}_{60A} = -t_3\mathbf{X}_{6+} + \mathbf{X}'_{60A} \;\; \lambda = 0$$

$$\mathbf{X}_{60B} = -t_4\mathbf{X}_{6+} + \mathbf{X}'_{60B}, \;\; \lambda = 0$$

$$\mathbf{X}_{60C} = -t_5\mathbf{X}_{6+} + \mathbf{X}'_{60C}, \;\; \lambda = 0$$

$$\mathbf{X}_{60D} = -t_6\mathbf{X}_{6+} + \mathbf{X}'_{60D}, \;\; \lambda = 0\,.$$

For clarity, this evolved solution is expanded out in full, block eigenvector form as follows:

(5.3)

$$\mathbf{X}_{6+} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix}, \;\; \lambda = C$$

$$\mathbf{X}_{6-} = -(t_6^2 + t_5^2 + t_4^2 + t_3^2)\begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} +$$

$$+ 2t_3\begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}'_{30} \end{pmatrix} + 2t_4\begin{pmatrix} 0 \\ 0 \\ C \\ \mathbf{0}_3 \end{pmatrix} + 2t_5\begin{pmatrix} 0 \\ C \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + 2t_6\begin{pmatrix} C \\ 0 \\ 0 \\ \mathbf{0}_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}'_{3-} \end{pmatrix}, \;\; \lambda = -C$$

$$\mathbf{X}_{60A} = -t_3\begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}'_{30} \end{pmatrix}, \;\; \lambda = 0$$

$$\mathbf{X}_{60B} = -t_4\begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ C \\ \mathbf{0}_3 \end{pmatrix}, \;\; \lambda = 0$$

$$\mathbf{X}_{60C} = -t_5\begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} 0 \\ C \\ 0 \\ \mathbf{0}_3 \end{pmatrix}, \;\; \lambda = 0$$

$$\mathbf{X}_{60D} = -t_6 \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mathbf{X}_{3+} \end{pmatrix} + \begin{pmatrix} C \\ 0 \\ 0 \\ \mathbf{0}_3 \end{pmatrix}, \ \lambda = 0 \, .$$

The URM6 Unity Root Matrix

By virtue of the eigenvalues, $\lambda = \pm C, 0,0,0,0$, the URM6 unity root matrix $\mathbf{A}_{60}$ has the same spectral decomposition [13] as its URM3 counterpart (2.40), i.e.

(5.4) $$\mathbf{A}_{60} = \frac{1}{2C}\left(\mathbf{X}_{6+}\mathbf{X}^{6+} - \mathbf{X}_{6-}\mathbf{X}^{6-}\right),$$

which is valid for all arbitrary, non-zero temporal parameters, t_3, t_4, t_5 and t_6, using the above, evolved forms of the eigenvectors (5.3).

For the full, six-quark solution, the matrix operator for the axis component of isospin ($\mathbf{I}_3$) is represented in URMT by the URM6, 6x6 zero matrix $\mathbf{A}_{60}$. This matrix is also specified below, in block matrix form, in terms of the fundamental, 3x3, unity root matrix $\mathbf{A}_0$, which is now relabelled $\mathbf{A}_{30}$ and explicitly written as an evolved function of time t_3 i.e. $\mathbf{A}_{30}(t_3)$ (4.3), reproduced below. The matrix $\mathbf{A}_{60}$ also embeds the URM3 eigenvectors $\mathbf{X}_+$ and $\mathbf{X}^-$, now relabelled $\mathbf{X}_{3+}$, $\mathbf{X}^{3-}$, plus the three additional integer parameters t_4, t_5, t_6

(5.5) $$\mathbf{A}_{60} = \begin{pmatrix} 0 & 0 & 0 & -t_6\mathbf{X}^{3-} \\ 0 & 0 & 0 & -t_5\mathbf{X}^{3-} \\ 0 & 0 & 0 & -t_4\mathbf{X}^{3-} \\ t_6\mathbf{X}_{3+} & t_5\mathbf{X}_{3+} & t_4\mathbf{X}_{3+} & \mathbf{A}_{30}(t_3) \end{pmatrix}, \ t_3, \ t_4, \ t_5, \ t_6 \in \mathbb{Z},$$

where

$\mathbf{A}_{30}(t_3) = -t_3\mathbf{A}_{3+} + \mathbf{A}'_{30}$ (4.3),
$\mathbf{A}_{3+} = \mathbf{A}'_{3+}$, invariant,
$\mathbf{A}'_{30} = \mathbf{A}_{30}(t_3 = 0)$, initial state

Notes

If t_4, t_5, t_6 are all zero then the eigenvector solution (5.3) reduces to the URM3 time-domain form (4.2).

Naming $\mathbf{A}_{60}$ a ‘unity root’ matrix is a slight misnomer in that it is really now only the embedded URM3 matrix $\mathbf{A}_{30}$ that contains true unity roots P, Q, R (2.5) – all other elements, i.e. those of embedded eigenvectors $\mathbf{X}_{3+}$, $\mathbf{X}^{3-}$ are not unity roots.

The URM6 eigenvector equations for $\mathbf{A}_{60}$ are as per URM3 (2.7), but now with four, zero eigenvector equations instead of one, i.e.

(5.6)

$$\mathbf{A}_{60}\mathbf{X}_{6+} = C\mathbf{X}_{6+},\ \mathbf{A}_{60}\mathbf{X}_{6-} = -C\mathbf{X}_{6-}$$

$$\mathbf{A}_{60}\mathbf{X}_{60A} = 0,\ \mathbf{A}_{60}\mathbf{X}_{60B} = 0,\ \mathbf{A}_{60}\mathbf{X}_{60C} = 0,\ \mathbf{A}_{60}\mathbf{X}_{60D} = 0.$$

The above, fully expanded, parametric form (5.3) emphasises that the original URM3 solution is now a URM6 solution, with four temporal parameters (t_3, t_4, t_5, t_6). The frequency domain solution also has an additional four frequency parameters (f_3, f_4, f_5, f_6) and, together with the original URM3 analytic solution, parameterised by integers *k* and *l*, (4.1), gives ten free parameters to play with.

Whilst the solution may appear sparse, i.e. numerous zero elements, this is illusory due to the fact that the variation has only been given in the time-domain. By also varying the frequency domain parameters, this sparsity disappears – see the full example solution in [1] for non-zero frequency variation.

The above general eigenvector solution, as per the URM3 solution, does not form an orthogonal basis for any arbitrary time $t_k, k = 3..6$, neither are the vectors of unit magnitude (most definitely not in fact), but their inner products relations remain invariant with time, hence they are unitary. There is a special solution [14], which starts out as orthogonal for zero t_k, but is not orthogonal for non-zero t_k. However, orthogonality is immaterial because the vectors are orthogonal to their reciprocal counterparts, as given by the inner products (2.26) – replace $\mathbf{X}_0$ (URM3) with $\mathbf{X}_{60A}$, $\mathbf{X}_{60B}$, $\mathbf{X}_{60C}$ and $\mathbf{X}_{60D}$ (URM6). That said, all vectors are linearly independent in both the standard (column vector) and reciprocal (row vector) bases, regardless of their evolved state, and can thus always form a basis spanning a six-dimensional vector space.

URM6 Conjugate Eigenvectors

The reciprocal (or conjugate) eigenvectors are defined exactly as per all URMT reciprocal vectors via the conjugate relations (2.22), where the **T** operator is now the following 6x6 matrix:

(5.7) $$\mathbf{T}_6 = \mathbf{T}^6 = \begin{pmatrix} \mathbf{I}_5 & 0 \\ 0 & -1 \end{pmatrix}.$$

(5.8)

$$\mathbf{X}^{6-} = (\mathbf{T}\mathbf{X}_{6+})^T,\ \mathbf{X}^{6+} = (\mathbf{T}\mathbf{X}_{6-})^T,$$

$$\mathbf{X}^{60A} = (\mathbf{T}\mathbf{X}_{60A})^T,\ \mathbf{X}^{60B} = (\mathbf{T}\mathbf{X}_{60B})^T,\ \mathbf{X}^{60C} = (\mathbf{T}\mathbf{X}_{60C})^T,\ \mathbf{X}^{60D} = (\mathbf{T}\mathbf{X}_{60D})^T$$

Note that the **T** operator still retains its Minkowski form (2.21).

As per URM3 (and all URMT incarnations), the zero eigenvectors are self-conjugate, whereas the plus and minus eigenvectors swap their subscript/superscript signage to reflect their eigenvalues to $\mathbf{A}_{60}$ when going from standard to conjugate form and vice versa.

Six Quark Eigenvectors

The six quarks are represented in terms of the URM6 eigenvector solution as follows, which, for the up, down and strange quarks, is the same as URM3 (2.15) barring the extension to six-element vectors:

(5.9)

$$\mathbf{u} = \frac{\mathbf{X}_{6+}}{\sqrt{2}C} \text{ up, } \mathbf{d} = \frac{\mathbf{X}_{6-}}{\sqrt{2}C} \text{ down,}$$
$$\mathbf{s} = \frac{\mathbf{X}_{60A}}{C} \text{ strange, } \mathbf{c} = \frac{\mathbf{X}_{60C}}{C} \text{ charm,}$$
$$\mathbf{b} = \frac{\mathbf{X}_{60B}}{C} \text{ bottom, } \mathbf{t} = \frac{\mathbf{X}_{60D}}{C} \text{ top.}$$

The above ordering $\mathbf{u},\mathbf{d},\mathbf{s},\mathbf{c},\mathbf{b},\mathbf{t}$ is given in terms of mass, with the lightest two quarks, i.e. the up and down quarks, given first, and the heaviest, top quark, last. However, the assignment of which zero isospin quark, **s**, **c**, **t** or **b,** to which zero eigenvector, $\mathbf{X}_{60A}$ to $\mathbf{X}_{60D}$, is arbitrary, and has been chosen such that the charm quark is assigned to $\mathbf{X}_{60C}$ merely because its eigenvector subscript contains the letter 'c', likewise for the bottom quark assigned to $\mathbf{X}_{60B}$. The strange quark is assigned to its URM3 equivalent (2.15), thus leaving the top quark assigned to $\mathbf{X}_{60D}$.

The anti-quarks are assigned to the URM6 reciprocal eigenvectors as follows, and also compatible with the URM3 assignments (2.29) of the up down and strange quarks:

(5.10)

$$\bar{\mathbf{u}} = \frac{\mathbf{X}^{6+}}{\sqrt{2}C}, \ \bar{\mathbf{d}} = \frac{\mathbf{X}^{6-}}{\sqrt{2}C}$$
$$\bar{\mathbf{s}} = \frac{\mathbf{X}^{60A}}{C}, \ \bar{\mathbf{c}} = \frac{\mathbf{X}^{60C}}{C}$$
$$\bar{\mathbf{b}} = \frac{\mathbf{X}^{60B}}{C}, \ \bar{\mathbf{t}} = \frac{\mathbf{X}^{60D}}{C}.$$

The zero isospin quarks, **s**, **c**,**t** and **b** are self-conjugate (URMT conjugate that is), unlike the up and down quarks, where $\bar{\mathbf{u}}$ is the URMT conjugate of the down quark, and $\bar{\mathbf{d}}$ is the URMT-conjugate of the up quark – note that the URMT conjugate (2.22) is not exactly the same as the Hermitian conjugate, explained in [1].

Inner Products

With six, linearly independent eigenvectors (quarks) and their reciprocals (anti-quarks), there are 36 possible inner products. However, they reduce very simply to their URM3, invariant forms and are absolutely invariant to all time and frequency variations. All the existing URM3, three-quark relations remain valid when the eigenvectors $\mathbf{X}_+$ and $\mathbf{X}_-$ are replaced by $\mathbf{X}_{6+}$ and $\mathbf{X}_{6-}$, and $\mathbf{X}_0$ is replaced by any one of $\mathbf{X}_{60A}$, $\mathbf{X}_{60B}$, $\mathbf{X}_{60C}$ or $\mathbf{X}_{60D}$.

The existing up and down quarks are Pythagorean sextuplets, i.e.

(5.11) $\mathbf{X}^{6+}\mathbf{X}_{6-}=0 \sim \bar{\mathbf{u}}\mathbf{d}=0$, $\mathbf{X}^{6-}\mathbf{X}_{6+}=0 \sim \bar{\mathbf{d}}\mathbf{u}=0$, Pythagoras

and the strange, charm top and bottom quarks satisfy the six-dimensional form of the hyperbolic DCE (2.26), i.e.

(5.12)
$\mathbf{X}^{60A}\mathbf{X}_{60A}=C^2 \sim \bar{\mathbf{s}}\mathbf{s}=1$, $\mathbf{X}^{60C}\mathbf{X}_{60C}=C^2 \sim \bar{\mathbf{c}}\mathbf{c}=1$, (2.26)
$\mathbf{X}^{60B}\mathbf{X}_{60B}=C^2 \sim \bar{\mathbf{b}}\mathbf{b}=1$, $\mathbf{X}^{60D}\mathbf{X}_{60D}=C^2 \sim \bar{\mathbf{t}}\mathbf{t}=1$.

The only other, two, non-zero inner products are

(5.13) $\mathbf{X}^{6+}\mathbf{X}_{6+}=2C^2 \sim \bar{\mathbf{u}}\mathbf{u}=1$, $\mathbf{X}^{6-}\mathbf{X}_{6-}=2C^2 \sim \bar{\mathbf{d}}\mathbf{d}=1$, (2.26).

and all other inner products are zero.

In quark-equivalent terms, the equations above amount to straightforward normalisation and orthogonality, i.e. for quarks $\mathbf{q},\mathbf{v} \in \{\mathbf{u},\mathbf{d},\mathbf{s},\mathbf{c},\mathbf{b},\mathbf{t}\}$ then

(5.14) $\bar{\mathbf{q}}\mathbf{v}=1$, for $\mathbf{q}=\mathbf{v}$ normalisation : $\bar{\mathbf{q}}\mathbf{v}=0$ for $\mathbf{q}\neq\mathbf{v}$ orthogonality.

The invariance of these inner products to variations in time and frequency domain variation is significant because it means that, whatever the particular value of the eigenvectors, unitarity, i.e. preservation of the inner product, is retained – a feature absolutely essential in QM to preserve probability.

A Composite Zero Eigenvector

The above quark assignment has a '2+4' nature to it in that the four, **s**, **c**, **b** and **t** quarks occupy a zero eigenvector subspace (zero isospin), whilst the two, up and down quarks have non-zero isospin, as represented by the plus and minus eigenvectors respectively. In URMT terms, the zero eigenvectors are hyperbolic in nature, whilst the plus and minus eigenvectors are Pythagorean, which means that URMT gives a structurally different representation of the zero isospin quarks, i.e. they do not satisfy the same inner product relations. Since the zero eigenvectors all satisfy the same, six-dimensional, hyperbolic DCE (5.12), there is a case to combine all four zero eigenvectors, $\mathbf{X}_{60A}$ to $\mathbf{X}_{60D}$, into a single, zero eigenvector $\mathbf{X}_{60}$ so that URM6 then effectively possesses just three eigenvectors, i.e. the standard plus, zero and minus base eigenvectors $\mathbf{X}_{6+}$, $\mathbf{X}_{60}$ and $\mathbf{X}_{6-}$. This composite $\mathbf{X}_{60}$ is formed from the following sum:

(5.15) $\mathbf{X}_{60}=\frac{1}{2}\left(\mathbf{X}_{60A}+\mathbf{X}_{60B}+\mathbf{X}_{60C}+\mathbf{X}_{60D}\right)$, $\mathbf{X}^{60}=(\mathbf{T}\mathbf{X}_{60})^T$.

Given this form of $\mathbf{X}_{60}$, it is easily verified, using the individual relations (5.6), that it satisfies the zero eigenvector, defining relation, i.e.

(5.16) $\mathbf{A}_{60}\mathbf{X}_{60} = 0$, $\mathbf{X}_{60} \sim \frac{1}{2}(\mathbf{s}+\mathbf{c}+\mathbf{b}+\mathbf{t})$, $\mathbf{A}_{60} \sim \mathbf{I}_3$,

and, in fact, every original URM3 eigenvector and inner product relation (2.26) is also satisfied. This means that the solution reverts to looking like URM3, even though it is cast in the 6x6 matrix form of URM6, i.e. what is a six-dimensional representation can be recast in the original, URM3, three-dimensional form. The sub-space of zero eigenvectors being that of the zero isospin, strange, charm, top and bottom quarks. Simply stated, the URM6 model becomes one of the up and down quarks, plus the four others lumped together as a single, zero isospin quark, which is in accord with the special place that the up and down quarks hold in our stable world.

It should be noted that some care must be exercised using $\mathbf{X}_{60}$ since evolving this composite, with the standard URM3-like evolution equation (4.2), is not quite the same as evolving the individual zero eigenvectors $\mathbf{X}_{60A}$ to $\mathbf{X}_{60D}$, as per (5.2), and then forming a composite $\mathbf{X}_{60}$. This point is expanded upon in [1], but it is always best to evolve the individual, zero eigenvectors $\mathbf{X}_{60A}$ etc. and then combine them.

Given this URM6 scheme can be reduced to just three, URM3-like eigenvectors, i.e. $\mathbf{X}_+$, $\mathbf{X}_0$, $\mathbf{X}_-$ (using the composite form $\mathbf{X}_{60} \sim \mathbf{X}_0$), then the '6' subscript can be dropped from all equations and they then revert to their URM3 form with a single subscript plus, zero or minus. Of course, this is all just in terms of the base set $\{\mathbf{X}_+, \mathbf{X}_0, \mathbf{X}_-\}$ of eigenvectors, and URM6 is three-fold degenerate, just like URM3, with the second and third difference and sum sets, $\{\mathbf{X}_{D+}, \mathbf{X}_{D0}, \mathbf{X}_{D-}\}$ and $\{\mathbf{X}_{S+}, \mathbf{X}_{S0}, \mathbf{X}_{S-}\}$ respectively, defined exactly as before, Section (3), in terms of the base set. Any one of the three sets could equally be used for a six-quark solution and thus, once again, suggestive of three quark colours.

Raising and Lowering Operators

There are the two raising and lowering matrices, $\mathbf{A}_{6+}$ and $\mathbf{A}_{6-}$, that can also be written exactly as per URM3 in their eigenvector forms given earlier (2.40) and (2.41). However, much the same comments apply here as for URM3, and readers are referred to [1] for more URM6/SU(6) specific details.

Compactification

Compactification, herein, is the reduction of the six-quark, six-dimensional solution to look like a single, three-dimensional quark.

The $\mathbf{X}_{6-}$ solution (5.2) is quadratic in all four temporal evolutionary parameters t_k, for the URM3, $\mathbf{X}_{6+}$ component only, whilst the other terms are linear or constant in time. This means that, for large evolutionary times $t_k >> 0$, all the evolved vectors align with $\mathbf{X}_{6+}$ as in

(5.17)
$\mathbf{X}_{6-} \approx -t^2\mathbf{X}_{6+} \sim \mathbf{d} \approx -t^2\mathbf{u}$, $|t| >> 0$, $t^2 = t_6^2 + t_5^2 + t_4^2 + t_3^2$,

$$\mathbf{X}_{60} \approx -t\mathbf{X}_{6+} \sim \mathbf{s} \approx -t_3\sqrt{2}\mathbf{u}\,,\ \mathbf{c} \approx -t_4\sqrt{2}\mathbf{u}\,,\ \mathbf{b} \approx -t_5\sqrt{2}\mathbf{u}\,,\ \mathbf{t} \approx -t_6\sqrt{2}\mathbf{u}\,,$$

i.e. the quarks align with the up quark so that, barring the evolutionary scale factor, i.e. disregarding scale, the quarks tend to look like the up quark.

In the frequency domain, for large frequencies, the situation is reversed and the quarks align with the down quark $\mathbf{X}_{6-}$.

In effect, what is a six-dimensional solution (four time-domain parameters and integers *k*,*l* (4.1)) appears, over time, to become just a two-dimensional solution (parameterised by *k*,*l*), i.e. that of the up quark $\mathbf{X}_{6+}$ (5.9), which is basically just the two-parameter, Pythagorean triple represented by the URM3, invariant eigenvector $\mathbf{X}_{3+}$ (2.2), hence a two-dimensional space (think of a vector pointing in three dimensional space, characterised by two angles, but arbitrary magnitude – eigenvectors are arbitrary to within a scale factor, which is t^2 here for large *t*. Thus, the solution exhibits the geometric property of compactification, i.e. the apparent shrinkage of higher dimensions with respect to the lower dimensions [14]. In other words, for large variations in the time or frequency domain, the six-quark solution can be made to look like either the up or down quark (but not both at once), to within a scale factor.

A three-fold degeneracy – quark families

A three-fold degeneracy (base, difference and sum-sets) has already been mentioned in Section (3) as suggestive of the quark colour degeneracy (RGB), but there is also a three-fold degeneracy in having three quark families, whereby only the first family, comprising the up and down quarks, form the stable particles of nature, with the other two families (strange, charm), (bottom, top), forming the more unstable particles.

This degeneracy in quark families is actually implicit in the URM6 solution by virtue that it is six-dimensional, albeit with a three-dimensional feel to it, i.e. it is a URM3 solution lifted to six dimensions (URM6), but retaining the same general features of URM3, notably only two non-zero eigenvalues, $\pm C$ for the up and down quarks, with all four other eigenvalues zero, representing the other two degenerate families with zero isospin.

That URM3 has been used as the starting point, with three quarks, up, down and strange, and not just the fundamental, two-quark up and down generation, may seem at odds with these degeneracy claims, but what has been completely omitted from this paper is the 2x2 URMT solution 'URM2', as fully detailed in [1]. In fact, [1] starts with spin and URM2, comprising only the up and down quarks, but moves on to URM3 and three quarks as the URM2 parametric evolution of the eigenvectors runs into difficulties, or at least not in line with URMT's aesthetics (the evolved vectors are no longer Pythagorean 'doubles'). Nevertheless, the URM2 representation theory is a complete, three-axis spin solution, and it is only URMT aesthetics, plus the more important fact that there are most definitely at least three quarks, that makes URM3, and three quarks, the minimal solution.

Most importantly though, just like URM3 was lifted here to URM6, in fact URM2 can also be lifted to give URM3. In other words, it is possible to start with URM2, and the up and down quarks, and go all the way up to URM6 and six quarks, as detailed in [14].

All that said, it is notable that the URMT eigenvector split could be thought of as a '2+4' form, rather than '2+2+2'. There is also no reason given here to stop at three-generations, i.e. why are there only three generations, not four etc.? Of course, there may be such generations, albeit it is understood that there are strong arguments against more than three generations.

Quantum Charge Operators

So far, only isospin has been given the URMT-treatment as a conserved quantum charge (strictly speaking, it is the magnitude of isospin that is conserved only under the strong force) but there are, of course, other quantum charges, such as strangeness, represented as usual by matrix operators that gives the charge as an eigenvalue to the operator when acting on the quark state vector, and these last few sub-sections detail these operators and their URMT-equivalent operators forms.

Isospin

The SU(6) form of this operator $\mathbf{I}_3$, which gives the third component of isospin, is identical to that used for SU(3), as per (2.39), reproduced below,

$$\mathbf{I}_3 = \frac{1}{2}\left(\mathbf{u}\bar{\mathbf{u}} - \mathbf{d}\bar{\mathbf{d}}\right) \text{ (2.39).}$$

The anti-particle form $\bar{\mathbf{I}}_3$ is, by (2.28), just the negation, i.e.

$$\bar{\mathbf{I}}_3 = -\mathbf{I}_3 \text{ (2.28).}$$

The URM6 form of $\mathbf{I}_3$ is also as per the SU(3) form, i.e.

$$\mathbf{I}_3 = \frac{\mathbf{A}_{60}}{2C} \text{ (2.16).}$$

Strangeness

Strangeness is given by the strangeness operator $\mathbf{S}$, constructed from the outer product of the strange quark and its anti-quark, as in

$$\text{(5.18) } \mathbf{S} = -\mathbf{s}\bar{\mathbf{s}},$$

and the antiquark form is the usual negation, i.e.

$$\text{(5.19) } \bar{\mathbf{S}} = -\mathbf{S} = \mathbf{s}\bar{\mathbf{s}}.$$

Using definitions (2.15) and (2.29) then, in terms of the URMT eigenvectors, $\mathbf{S}$ and $\bar{\mathbf{S}}$ are defined as

$$\text{(5.20) } \mathbf{S} = -\frac{\mathbf{X}_{60A}\mathbf{X}^{60A}}{C^2},\ \bar{\mathbf{S}} = \frac{\mathbf{X}_{60A}\mathbf{X}^{60A}}{C^2}.$$

The action of **S** on the strange quark is given by

(5.21) $\mathbf{Ss} = (-\mathbf{s}\bar{\mathbf{s}})\mathbf{s} = -\mathbf{s}(\bar{\mathbf{s}}\mathbf{s})$,

and since $\bar{\mathbf{s}}\mathbf{s} = 1$ (2.32) then

(5.22) $\mathbf{Ss} = -\mathbf{s}$.

The eigenvalue is therefore -1, and the strangeness of the strange quark accordingly the same (negative by convention). Doing the same for the strange anti-quark gives an eigenvalue of +1, i.e. the strange anti-quark has a strangeness of +1 as in

(5.23) $\mathbf{S}\bar{\mathbf{s}} = \bar{\mathbf{s}}$.

Other quantum charges, defined in a similar way, are summarised below.

Charm

The charm operator is defined as

(5.24) $\mathbf{C} = \mathbf{c}\bar{\mathbf{c}}$, $\overline{\mathbf{C}} = -\mathbf{C} = -\mathbf{c}\bar{\mathbf{c}}$, $\mathbf{C} = \dfrac{\mathbf{X}_{60C}\mathbf{X}^{60C}}{C^2}$, $\overline{\mathbf{C}} = \dfrac{\mathbf{X}_{60C}\mathbf{X}^{60C}}{C^2}$

and the charge is positive for the charm quark, by convention, and negative for its anti-particle, i.e. the converse of the strange quark, with which it is paired in the second generation of quarks.

Bottom

The bottom operator is defined as follows, where its symbol **B** is intentionally superscripted here with a ‘q’ to differentiate it from the baryon number operator, (5.27) further below:

(5.25) $\mathbf{B}^q = -\mathbf{b}\bar{\mathbf{b}}$, $\overline{\mathbf{B}}^q = -\mathbf{B} = \mathbf{b}\bar{\mathbf{b}}$, $\mathbf{B}^q = -\dfrac{\mathbf{X}_{60B}\mathbf{X}^{60B}}{C^2}$, $\overline{\mathbf{B}}^q = \dfrac{\mathbf{X}_{60B}\mathbf{X}^{60B}}{C^2}$

The charge of the bottom quark is -1, as per convention, and its anti-quark has a charge of +1.

Top

The top charge operator is defined as follows, where its symbol $\mathbf{T}^q$ is intentionally superscripted here with a ‘q’ to differentiate it from the URMT **T** operator (5.7):

(5.26) $\mathbf{T}^q = \mathbf{t}\bar{\mathbf{t}}$, $\overline{\mathbf{T}}^q = -\mathbf{T}^q = -\mathbf{t}\bar{\mathbf{t}}$, $\mathbf{T}^q = \dfrac{\mathbf{X}_{60D}\mathbf{X}^{60D}}{C^2}$, $\overline{\mathbf{T}}^q = -\dfrac{\mathbf{X}_{60D}\mathbf{X}^{60D}}{C^2}$

The top charge is positive for the top quark, by convention, and negative for its anti-particle (as in $\bar{\mathbf{t}}\bar{\mathbf{T}}^{q} = -\bar{\mathbf{t}}$), and is intentionally the converse of the bottom quark, with which it is paired in the third generation of quarks.

Baryon Number

The baryon number (eigenvalue symbol *B*) of each of the six quarks is 1/3, and -1/3 for the anti-quarks, and therefore its operator, symbol **B**, is defined as

(5.27)

$$\mathbf{B} = \frac{1}{3}\left(\mathbf{u}\bar{\mathbf{u}} + \mathbf{d}\bar{\mathbf{d}} + \mathbf{s}\bar{\mathbf{s}} + \mathbf{c}\bar{\mathbf{c}} + \mathbf{b}\bar{\mathbf{b}} + \mathbf{t}\bar{\mathbf{t}}\right),\ \bar{\mathbf{B}} = -\mathbf{B},$$

$$\mathbf{B} = \frac{1}{6C^2}\left(\mathbf{X}_{6+}\mathbf{X}^{6+} + \mathbf{X}_{6-}\mathbf{X}^{6-}\right) + \frac{1}{3C^2}\left(\mathbf{X}_{60A}\mathbf{X}^{60A} + \mathbf{X}_{60B}\mathbf{X}^{60B}\right) + \frac{1}{3C^2}\left(\mathbf{X}_{60C}\mathbf{X}^{60C} + \mathbf{X}_{60D}\mathbf{X}^{60D}\right).$$

with $\bar{\mathbf{B}}$, accordingly, its negation.

Hypercharge operator

The hypercharge operator is defined in terms of the Baryon operator **B** (5.27), Strangeness operator **S** (5.20), Charm operator **C** (5.24), Bottom operator $\mathbf{B}^{q}$ (5.25) and top operator $\mathbf{T}^{q}$ (5.26):

(5.28) $\mathbf{Y} = \mathbf{B} + \mathbf{S} + \mathbf{C} + \mathbf{B}^{q} + \mathbf{T}^{q}$.

Using the above definitions for each of the individual operators, this expands in terms of the quark eigenvectors to

(5.29)

$$\mathbf{Y} = \frac{1}{3}\left[\mathbf{u}\bar{\mathbf{u}} + \mathbf{d}\bar{\mathbf{d}} - 2\left(\mathbf{s}\bar{\mathbf{s}} + \mathbf{b}\bar{\mathbf{b}}\right) + 4\left(\mathbf{c}\bar{\mathbf{c}} + \mathbf{t}\bar{\mathbf{t}}\right)\right],\ \bar{\mathbf{Y}} = -\mathbf{Y},$$

$$\mathbf{Y} = \frac{1}{3C^2}\left(\frac{\mathbf{X}_{6+}\mathbf{X}^{6+}}{2} + \frac{\mathbf{X}_{6-}\mathbf{X}^{6-}}{2}\right) - \frac{2}{3C^2}\left(\mathbf{X}_{60A}\mathbf{X}^{60A} + \mathbf{X}_{60B}\mathbf{X}^{60B}\right) + \frac{4}{3C^2}\left(\mathbf{X}_{60C}\mathbf{X}^{60C} + \mathbf{X}_{60D}\mathbf{X}^{60D}\right)$$

Electric Charge

The electric charge (eigenvalue symbol Q) of each of the three quarks, up, charm and strange (**u**,**c**,**t**), is 2/3, and -1/3 for the three quarks, down strange and bottom (**d**,**s**,**b**), with the signs inverted for their anti-quark forms, and therefore its operator, symbol **Q**, is defined as

(5.30)

$$\mathbf{Q} = \frac{1}{3}\left(2\mathbf{u}\bar{\mathbf{u}} + 2\mathbf{c}\bar{\mathbf{c}} + 2\mathbf{t}\bar{\mathbf{t}} - \mathbf{d}\bar{\mathbf{d}} - \mathbf{s}\bar{\mathbf{s}} - \mathbf{b}\bar{\mathbf{b}}\right),\ \bar{\mathbf{Q}} = -\mathbf{Q}\,.$$

$$\mathbf{Q} = \frac{1}{3C^2}\left(\mathbf{X}_{6+}\mathbf{X}^{6+} + 2\mathbf{X}_{60C}\mathbf{X}^{60C} + 2\mathbf{X}_{60D}\mathbf{X}^{60D}\right) - \frac{1}{6C^2}\left(\mathbf{X}_{6-}\mathbf{X}^{6-} + 2\mathbf{X}_{60A}\mathbf{X}^{60A} + 2\mathbf{X}_{60B}\mathbf{X}^{60B}\right).$$

The charge relates to the third component of isospin and hypercharge by

(5.31) $$Q = I_3 + \frac{Y}{2}.$$

This can be verified by substituting for the operator forms of $\mathbf{I}_3$ and $\mathbf{Y}$ in place of the eigenvalues, to get the operator $\mathbf{Q}$.

Quark Charges Summary Table

The following is a summary table of the six-quark charges, which can be verified by the action of the appropriate matrix operator upon the quark (eigenvector) to obtain its eigenvalue (the charge).

(5.32)

	I	I_3	Y	S	B	Q	C	B^q	T^q
u	½	½	⅓	0	⅓	⅔	0	0	0
d	½	-½	⅓	0	⅓	−⅓	0	0	0
s	0	0	−⅔	-1	⅓	−⅓	0	0	0
c	0	0	⅔	0	⅓	⅔	1	0	0
b	0	0	−⅔	0	⅓	−⅓	0	-1	0
t	0	0	⅔	0	⅓	⅔	0	0	1
$\bar{\mathbf{u}}$	½	-½	−⅓	0	−⅓	−⅔	0	0	0
$\bar{\mathbf{d}}$	½	½	−⅓	0	−⅓	⅓	0	0	0
$\bar{\mathbf{s}}$	0	0	⅔	1	−⅓	⅓	0	0	0
$\bar{\mathbf{c}}$	0	0	−⅔	0	−⅓	−⅔	-1	0	0

$\bar{\mathbf{b}}$	0	0	⅔	0	−⅓	⅓	0	1	0
$\bar{\mathbf{t}}$	0	0	−4⁄3	0	−⅓	−⅔	0	0	-1

6 A Numeric Example Six-Quark Solution

This example is taken directly from [1], and is an evolved solution, obtained from a simple URM3 solution by lifting and evolving with both non-zero time and frequency domain parameters. The numbers are purely illustrative, and actual values in this example are not ascribed any physical significance.

The solution is given for a unity eigenvalue $C = 1$.

$$(6.1)\quad \mathbf{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} -22 \\ -14 \\ -6 \\ -570 \\ 245 \\ 621 \end{pmatrix},\ \mathbf{d} = \frac{1}{\sqrt{2}}\begin{pmatrix} 6 \\ 4 \\ 2 \\ 172 \\ -73 \\ -187 \end{pmatrix},$$

$$\mathbf{s} = \begin{pmatrix} -6 \\ -4 \\ -2 \\ -179 \\ 77 \\ 195 \end{pmatrix},\ \mathbf{b} = \begin{pmatrix} -6 \\ -4 \\ -1 \\ -160 \\ 68 \\ 174 \end{pmatrix},\ \mathbf{c} = \begin{pmatrix} -6 \\ -3 \\ -2 \\ -148 \\ 63 \\ 161 \end{pmatrix},\ \mathbf{t} = \begin{pmatrix} -5 \\ -4 \\ -2 \\ -136 \\ 58 \\ 148 \end{pmatrix}.$$

Inspection of the elements of all six quark eigenvectors shows them to be unique with their own distinct elements. This means that, whilst they satisfy all the usual inner product relations (2.32), at the same time they have different vector magnitudes – all due to the choice of evolutionary parameters in the time and/or frequency domain. Choice of the actual parametric values thus dictates their magnitude, whilst keeping all inner products invariant, and thus it is suggested that the physical properties of the individual quarks, such as mass, can be tuned by appropriate choice of the parameters.

The anti-quarks are obtained from the conjugate eigenvectors as per (5.10), to give

(6.2)

$$\bar{\mathbf{u}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 6 & 4 & 2 & 172 & -73 & 187 \end{pmatrix}$$

$$\bar{\mathbf{d}} = \frac{1}{\sqrt{2}}\begin{pmatrix} -22 & -14 & -6 & -570 & 245 & -621 \end{pmatrix}$$

$$\bar{\mathbf{s}} = \begin{pmatrix} -6 & -4 & -2 & -179 & 77 & -195 \end{pmatrix}$$

$$\bar{\mathbf{c}} = \begin{pmatrix} -6 & -3 & -2 & -148 & 63 & -161 \end{pmatrix}$$

$$\bar{\mathbf{b}} = \begin{pmatrix} -6 & -4 & -1 & -160 & 68 & -174 \end{pmatrix}$$

$$\bar{\mathbf{t}} = \begin{pmatrix} -5 & -4 & -2 & -136 & 58 & -148 \end{pmatrix}.$$

The up and down quarks are Pythagorean sextuplets as in, for example, $\bar{\mathbf{d}}\mathbf{u} = 0$, which, using the numeric values above, gives

$$(6.3)\ \bar{\mathbf{d}}\mathbf{u} = \frac{1}{\sqrt{2}}\begin{pmatrix} -22 & -14 & -6 & -570 & 245 & -621 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} -22 \\ -14 \\ -6 \\ -570 \\ 245 \\ 621 \end{pmatrix} =$$

$$(-22)^2 + (-14)^2 + (-6)^2 + (-570)^2 + 245^2 - 621^2 = 0.$$

Likewise for the down and up anti-quark, inner product $\bar{\mathbf{u}}\mathbf{d} = 0$.

The other four quarks, i.e. the strange, charm, bottom and top, have hyperbolic inner products (5.12). For example, the strange and its anti-quark have the hyperbolic inner product $\bar{\mathbf{s}}\mathbf{s} = 1$, which, using the above numeric values, expands to

$$(6.5)\ \bar{\mathbf{s}}\mathbf{s} = \begin{pmatrix} -6 & -4 & -2 & -179 & 77 & -195 \end{pmatrix}\begin{pmatrix} -6 \\ -4 \\ -2 \\ -179 \\ 77 \\ 195 \end{pmatrix} =$$

$$(-6)^2 + (-4)^2 + (-2)^2 + (-179)^2 + 77^2 - 195^2 = 1^2.$$

Third component of Isospin

Using the definition (2.39) for $\mathbf{I}_3$, and the numeric values (6.1) and (6.2) for the quarks and anti-quarks respectively, then $\mathbf{I}_3$ evaluates to

$$(6.6)\ \mathbf{I}_3 = \frac{1}{2}\begin{pmatrix} 0 & -2 & -4 & -182 & 68 & -194 \\ 2 & 0 & -2 & -64 & 21 & -67 \\ 4 & 2 & 0 & 54 & -26 & 60 \\ 182 & 64 & -54 & 0 & -265 & 111 \\ -68 & -21 & 26 & 265 & 0 & 241 \\ -194 & -67 & 60 & 111 & 241 & 0 \end{pmatrix}.$$

Baryon number

Using the definition (5.27) for $\mathbf{B}$, and the numeric values (6.1) and (6.2) for the quarks and anti-quarks respectively, then **B** evaluates to

$$(6.7)\ \mathbf{B} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Hypercharge

Using the definition (5.29) for **Y,** and the numeric values (6.1) and (6.2) for the quarks and anti-quarks respectively, then **Y** evaluates to

$$(6.8)\ \mathbf{Y} = \frac{1}{3}\begin{pmatrix} -32 & -30 & 12 & -1398 & 606 & -1524 \\ -30 & -20 & 6 & -1104 & 477 & -1203 \\ 12 & 6 & 10 & 150 & -60 & 162 \\ -1398 & -1104 & 150 & -51722 & 22353 & -56367 \\ 606 & 477 & -60 & 22353 & -9659 & 24360 \\ 1524 & 1203 & -162 & 56367 & -24630 & 61429 \end{pmatrix}.$$

Electric charge

Using the definition (5.30) for **Q**, and the numeric values (6.1) and (6.2) for the quarks and anti-quarks respectively, then **Q** evaluates to

$$(6.9)\ \mathbf{Q} = \frac{1}{6}\begin{pmatrix} -32 & -36 & 0 & -1944 & 810 & -2106 \\ -24 & -20 & 0 & -1296 & 540 & -1404 \\ 24 & 12 & 10 & 312 & -138 & 342 \\ -852 & -912 & -12 & -51722 & 21558 & -56034 \\ 402 & 414 & 18 & 23148 & -9659 & 25083 \\ 942 & 1002 & 18 & 56700 & -23637 & 61429 \end{pmatrix}.$$

Finally, using these operators and eigenvectors, some of the charges given in the table (5.32) can be verified.

7 Summary and Conclusions

A comparison of the eigenvector algebra of URM3 with that of quantum mechanics, shows it to be equivalent to a three-state, isospin representation of the up, down and strange quarks. By analogy with the mathematically identical, quantum-mechanical treatment of angular momentum and/or spin, the existing URM3 incarnation has been expanded from a single-axis, to a three-axis, three-fold representation indicative of the red, green, blue colour representation of quarks.

The classic URMT evolution of the eigenvectors, attributed to the number-theoretic congruences satisfied by the dynamical variables of the unity root matrix, has been recast in an exponential, unitary operator form, which shows the eigenvector evolution to be the same as the time-dependent evolution of the wavefunction. This latter equivalence thus enabling a recast of URMT's founding Invariance Principle as a statement in the action of the dynamical variables forming the unity root matrix.

By the URMT method of lifting solutions to higher dimensions, the URM3, three-quark scheme has been expanded to encompass all six known quarks in a 6x6, URM6 matrix operator representation. This URM6 scheme naturally places the up and down quarks in a special category, whereby their eigenvector inner products satisfy Pythagoras, whilst the quarks in the second and third redundant generations satisfy hyperbolic Diophantine integer equations, thus distinguishing the first quark generation of up and down quarks (forming stable nucleons), as structurally different from the second and third generations (that form unstable particles).

From the results presented, it is evident that an integer representation of the quark flavour model can be derived purely from number theory, with no recourse to the foundations of classical quantum mechanics.

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