Unity Root Matrix Theory
and the Riemann Hypothesis

Richard J. Miller, http://www.urmt.org
Email: richard@microscitech.com

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Abstract

This short article shows how Unity Root Matrix Theory (URMT), [1] and [2], naturally provides eigenvalue solutions Z in the complex plane, located at $Z = 1/2 \pm ib$, for an infinite set of real b, i.e. they lie along the same line as the roots of the non-trivial zeros in the Riemann Hypothesis. The work links these eigenvalues to an integer eigenvector equation with related nth order congruence relations, unity roots and Diophantine equations, and is an example of linking physics in integers, via URMT, with the Riemann Hypothesis.

Document Status: This work was first published in draft form in June 2011 (Draft 08/06/2011). It has been edited since then, with a few more numbers, but all equations and results remain the same. Since 2011, URMT itself has moved on considerably, having been extended from a 3x3 matrix formulation to higher order matrices and real physical applications; see the web-site www.urmt.org [5] for more details. This article is to appear in a forthcoming book detailing URMT and all its links with number theory, due for publication mid 2013.

Acronyms and Abbreviations

DCE : Dynamical Conservation Equation
FLT : Fermat's Last Theorem
URMT : Unity Root Matrix Theory

References

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Background

The relation of the eigenvalues of a Hermitian operator to the non-trivial zeros in the Riemann Hypothesis is covered by the Hilbert Polya conjecture, see [3]. The article in [3] mentions a refinement called 'the Berry-Keating conjecture' 1999, which speculates on a possible Hamiltonian operator – see [3] for the citation. A more recent paper [4] on arXiv.org studies this Hamiltonian further.

The work in this article also studies the eigenvalues of a matrix operator which, itself, is Hermitian-like, see [1], and relates the eigenvalues to the non-trivial Riemann zeros. However, the approach is markedly different, employing the methods of URMT, [1] and [2], and working entirely in integers, but not without some intriguing similarities to existing work in number theory and the discrete formulation of physical laws. For example, the matrix operator has an associated 'Dynamical Conservation Equation' (the DCE [1]), likened to an energy conservation equation and formulated in terms of dynamical variables (velocity or momentum per unit mass), which, themselves, are defined as nth order, integer unity roots, isomorphic with the complex roots of unity, thereby linking to Hermitian operators and Hamiltonians.

Introduction

In brief, the work presented herein shows that the two complex roots of a special characteristic equation (the Dynamical Conservation Equation in [1]), are of the form $Z = 1/2 \pm ib$, for an infinite set of real b, every member related to the other, single integer root of +1, which is invariant for all b, hence all complex roots lie on the line $\mathrm{Re}(Z) = 1/2$.

A numeric, cubic example is given at the end.

The work actually starts with the case $\mathrm{Re}(Z) = -1/2$ (with an associated eigenvalue $\lambda = +1$) and moves on to $\mathrm{Re}(Z) = +1/2$ (associated eigenvalue $\lambda = -1$). This is only because URMT is formulated for an invariant, unity eigenvalue of $\lambda = +1$, and then extended to arbitrary eigenvalue C, including the aforementioned case where $C = -1$.

Very briefly...

For a 3x3 Integer matrix $\mathbf{A}$, eigenvector $\mathbf{X}$, with three unique eigenvalues λ_1, λ_2, λ_3, by defining one of them (λ_1) to be of unit magnitude, e.g. $\lambda_1 = \pm 1$, with eigenvector equation

$$\mathbf{AX} = \lambda_1 \mathbf{X},$$

and with the other two eigenvalues (λ_2, λ_3) of the following complex form:

$$\lambda_2 = a + bi,\ \lambda_3 = a - bi,\ \text{i.e. } \lambda_3 = \lambda_2^*,\ a, b \in \mathbb{R},\ b \neq 0$$

then, if the trace of $\mathbf{A}$ is zero, the sum of the eigenvalues is zero, i.e.

$$\lambda_1 + \lambda_2 + \lambda_3 = 0,$$

and for $\lambda_1 = \mp 1$ this implies

$$2\,\mathrm{Re}(\lambda_2, \lambda_3) = \pm 1$$

i.e.

$$a = \pm \frac{1}{2}\ \forall b \in \mathbb{R}.$$

By applying a parameterised transformation to matrix $\mathbf{A}$, that preserves the zero trace, the unity eigenvalue λ_1 and its associated eigenvector $\mathbf{X}$ can remain invariant, whilst the value for b varies over an infinite range as the transformation parameters vary over an infinite set. Hence all complex roots lie on the line $\mathrm{Re}(\lambda_2, \lambda_3) = \pm 1/2$.

Details

All variables are integers except b, which is real. Some later relaxation in this assumption is possible, e.g. the variational parameters η, δ and ε (used later) may also be real if desired. The only other exception is that the variable a, used above, is replaced with its rational value of $\pm 1/2$.

The 3x3 'Unity Root Matrix' $\mathbf{A}$, comprising integer elements P, Q, R and their conjugate forms $\overline{P}, \overline{Q}, \overline{R}$, is defined as follows

$$\mathbf{A} = \begin{pmatrix} 0 & R & \overline{Q} \\ \overline{R} & 0 & P \\ Q & \overline{P} & 0 \end{pmatrix},$$

$$P, Q, R, \overline{P}, \overline{Q}, \overline{R} \in \mathbb{Z},$$
$$(P, Q, R) \neq (0,0,0), \ (\overline{P}, \overline{Q}, \overline{R}) \neq (0,0,0)$$

Note that $\mathbf{A}$ has a zero lead diagonal and is therefore traceless, hence the sum of its eigenvalues is zero.

The eigenvector $\mathbf{X}$, for eigenvalue λ_1, comprises integer coordinates x, y, z, and is defined by

$$\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \ x, y, z \in \mathbb{Z}, \ (x, y, z) \neq (0,0,0)$$

The coordinates x, y, z will later be seen to satisfy an nth order Diophantine equation.

The matrix elements $P, Q, R, \overline{P}, \overline{Q}, \overline{R}$, of $\mathbf{A}$, are termed 'dynamical variables' in [1], and relate to the coordinates x, y, z via the following congruences, for some integer exponent $n >= 2$, (this fact will be of relevance near the end)

$$P^n \equiv +1 (\bmod x), \ \overline{P}^n \equiv +1 (\bmod x)$$
$$Q^n \equiv +1 (\bmod y), \ \overline{Q}^n \equiv +1 (\bmod y)$$
$$R^n \equiv -1 (\bmod z), \ \overline{R}^n \equiv -1 (\bmod z).$$

$\overline{P}, \overline{Q}, \overline{R}$ are conjugates of P, Q, R and are related to each other by 'Conjugate relations' [1]:

$$\overline{P} \equiv P^{n-1} \pmod{x},$$
$$\overline{Q} \equiv Q^{n-1} \pmod{y},$$
$$\overline{R} \equiv -R^{n-1} \pmod{z}.$$

The above defining properties of the dynamical variables make them isomorphic with the complex roots of unity, and thus make the matrix $\mathbf{A}$ Hermitian-like, i.e. it is equal to its transpose conjugate, to within a multiple of the coordinate modulus.

A kinetic term K is an defined as

$$K = P\overline{P} + Q\overline{Q} + R\overline{R},$$

and a potential term V is defined as

$$V = PQR + \overline{P}\overline{Q}\overline{R}.$$

Note that the potential is also the determinant of $\mathbf{A}$, i.e. $\det(\mathbf{A}) = V = PQR + \overline{P}\overline{Q}\overline{R}$.

The following eigenvector equation, eigenvalue λ, is now solved for all three eigenvalues, with one of the eigenvalues constrained to unity by definition

$$\mathbf{AX} = \lambda \mathbf{X}.$$

Using the potential term V and kinetic term K, the characteristic equation for $\mathbf{A}$ can be written as

$$-\lambda^3 + K\lambda + V = 0.$$

For unity eigenvalue, $\lambda = 1$, this characteristic equation satisfies the following Dynamical Conservation Equation (the DCE [1])

$$+1 = K + V, \ \lambda = +1,$$

and the characteristic equation factors as follows

$$(-\lambda + 1)(\lambda^2 + \lambda + V) = 0.$$

Note that, for reference further below, if $\lambda = -1$ then the DCE becomes $-K + V = -1$.

Denoting the three roots as λ_1, λ_2, λ_3 then

$$\lambda_1 = +1 \text{ (by definition)},$$

and the other two roots λ_2, λ_3 are given by the familiar quadratic solution

$$\lambda_2, \lambda_3 = -\frac{1}{2} \pm \frac{\sqrt{(1-4V)}}{2}.$$

Assuming the potential is greater than unity, i.e. $V > 1$, which can always be satisfied by a local transformation (see further below), then the two roots λ_2 and λ_3 are complex and λ_3 is the conjugate of λ_2.

Thus, for some real value b given by,

$$b = \frac{\sqrt{(4V-1)}}{2},\ V \geq 1,\ b \in \mathbb{R}.$$

the two complex roots are of the form

$$\lambda_2 = -\frac{1}{2} + ib,\ \lambda_3 = -\frac{1}{2} - ib.$$

Because the trace of $\mathbf{A}$ is zero, the sum of the eigenvalues is zero, i.e.

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 - \frac{1}{2} - \frac{1}{2} = 0.$$

These complex eigenvectors have a real part of $-1/2$. To convert to a real part of +1/2, the sign of the matrix $\mathbf{A}$ is reversed from + to -, and so too its eigenvalue. The eigenvector equation is now for matrix $-\mathbf{A}$ and eigenvalue $\lambda_1 = -1$, instead of $+\mathbf{A}$ and $\lambda_1 = +1$, as in

$$(-\mathbf{A})\mathbf{X} = -\mathbf{X}.$$

For the same eigenvector $\mathbf{X}$, the characteristic equation now factors as

$$(\lambda + 1)(\lambda^2 - \lambda - V) = 0,$$

and has the three roots λ_1, λ_2, λ_3, as follows, where b is now calculated as

$$b = \frac{\sqrt{(-1-4V)}}{2}, \; V \le -1,$$

$$\lambda_1 = -1, \; \lambda_2 = +\frac{1}{2} + ib, \; \lambda_3 = +\frac{1}{2} - ib.$$

Note that V also flips sign by virtue of all the dynamical variables P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ changing sign as $\mathbf{A} \to -\mathbf{A}$, so that b is now calculated as above. With $\lambda_1 = -1$ the DCE also changes to

$$-K + V = -1, \; \lambda_1 = -1.$$

The sum of the eigenvalues is again zero, and so now

$$\lambda_1 + \lambda_2 + \lambda_3 = -1 + \frac{1}{2} + \frac{1}{2} = 0.$$

The important bit

So far, this problem defines one solution for λ_2 and λ_3 associated with the single eigenvalue λ_1, eigenvector $\mathbf{X}$. But the matrix $\mathbf{A}$ can be transformed by three local integer variations, η, δ and ε, to obtain an infinite set of complex eigenvalues, λ_2 and λ_3, whilst preserving λ_1 and $\mathbf{X}$ as follows:

Consider the following transformation matrix $\mathbf{\Delta}$ defined as

$$\mathbf{\Delta} = \begin{pmatrix} 0 & +\eta z & -\eta y \\ -\delta z & 0 & +\delta x \\ +\varepsilon y & -\varepsilon x & 0 \end{pmatrix},$$

with an all-zero lead diagonal making it traceless as for $\mathbf{A}$.

This matrix $\mathbf{\Delta}$, acting on eigenvector $\mathbf{X}$, is such that $\mathbf{\Delta X} = 0$, and so the original eigenvector problem can be re-written in terms of both $\mathbf{A}$ and $\mathbf{\Delta}$ as

$$(\mathbf{A} + \mathbf{\Delta})\mathbf{X} = \mathbf{AX} + \mathbf{\Delta X} = \mathbf{AX} = \mathbf{X}.$$

The transformed matrix $\mathbf{A} \to \mathbf{A} + \mathbf{\Delta}$ thus has elements (dynamical variables), which transform as follows

$$P \to P + \delta x, \; \overline{P} \to \overline{P} - \varepsilon x$$
$$Q \to Q + \varepsilon y, \; \overline{Q} \to \overline{Q} - \eta y$$
$$R \to R + \eta z, \; \overline{R} \to \overline{R} - \delta z.$$

and so the potential V transforms as

$$V = (PQR + \overline{P}\overline{Q}\overline{R}) \to (P + \delta x)(Q + \varepsilon y)(R + \eta z) + (\overline{P} - \varepsilon x)(\overline{Q} - \eta y)(\overline{R} - \delta z) .$$

Note that the cubic variational term in $\delta\varepsilon\eta xyz$ cancels to zero. Thus, there are linear and quadratic variational terms only.

Whilst under the transformation $\mathbf{\Delta}$, there will always remain a single, invariant eigenvalue root λ_1 and associated invariant eigenvector $\mathbf{X}$ (since $(\mathbf{A} + \mathbf{\Delta})\mathbf{X} = \mathbf{AX}$ by the above), the potential V is not, in general, invariant, and will change with $\mathbf{\Delta}$. As a consequence, b will also change.

Given there are three possible, completely arbitrary, integer variations η, δ and ε to work with, then V can always be modified such that the roots λ_2 and λ_3 are complex and cover an infinite set of values for V, and therefore also b. However, the sum of the eigenvalues remains zero, and so the real part of the complex eigenvalues remains $-1/2$ (for $\lambda_1 = +1$) or $+1/2$ (for $\lambda_1 = -1$).

Note that an infinite set of values for V can always be obtained with associated complex eigenvalues since, for example, just by varying η, with $\delta = 0$ and $\varepsilon = 0$, then V transforms as follows

$$V \to V + \eta(PQz - \overline{P}\overline{R}y) ,$$

and given that the bracketed term $(PQz - \overline{P}\overline{R}y)$ is invariant, i.e. all its values are their initial values prior to transformation, then V is a simple straight-line function of the variation η. Consequently, with the right choice of sign for η, V can always be transformed to retain complex roots and, of course, η is completely arbitrary and spans the infinite set of integers. Lastly on this matter, the variations η, δ and ε can be real, not necessarily integer.

Summarising

The eigenvalue problem

$$\mathbf{AX} = \mathbf{X} \text{ or } (-\mathbf{A})\mathbf{X} = -\mathbf{X} ,$$

with fixed eigenvalue solution

$$\lambda_1 = +1 \text{ or } \lambda_1 = -1 ,$$

can be associated with an infinite family of complex roots, λ_2 and λ_3, for real b,

$$\lambda_2 = -\frac{1}{2} + ib,\ \lambda_3 = -\frac{1}{2} - ib, \text{ for } \lambda_1 = +1$$

or

$$\lambda_2 = +\frac{1}{2} + ib,\ \lambda_3 = +\frac{1}{2} - ib, \text{ for } \lambda_1 = -1,$$

i.e. all roots lie on the complex line

$$\mathrm{Re}(\lambda_2, \lambda_3) = -1/2,\ \lambda_1 = +1$$

or

$$\mathrm{Re}(\lambda_2, \lambda_3) = +1/2,\ \lambda_1 = -1$$

where $Cmplx(\lambda_2, \lambda_3) = b$, as given by

$$b = \frac{\sqrt{(4V-1)}}{2},\ V \geq +1,\ \lambda_1 = +1$$

or

$$b = \frac{\sqrt{(-1-4V)}}{2},\ V \leq -1,\ \lambda_1 = -1,$$

and the potential V is a function of dynamical variables ($P, Q, R, \overline{P}, \overline{Q}, \overline{R}$), which are now, themselves, a function of three arbitrary, variational parameters η, δ and ε.

Notes

1) If $\overline{P} = P$, $\overline{Q} = Q$, $\overline{R} = -R$ (termed 'Pythagoras Conditions' in [1]), then $V = 0$, $\lambda_1 = +1$, $\lambda_2 = 0$, $\lambda_3 = -1$, and the eigenvector $\mathbf{X}$ (for λ_1) is a Pythagorean triple, and so too is the eigenvector '$\mathbf{X}_-$' for eigenvalue $\lambda_3 = -1$, see [1]. The eigenvector '$\mathbf{X}_0$', for $\lambda_2 = 0$, satisfies the hyperbolic DCE:

$$+1 = P^2 + Q^2 - R^2.$$

2) With the dynamical variables P, Q, R and $\overline{P}, \overline{Q}, \overline{R}$ defined by the nth order congruences,

$$P^n \equiv +1 (\bmod x),\ \overline{P}^n \equiv +1 (\bmod x)$$
$$Q^n \equiv +1 (\bmod y),\ \overline{Q}^n \equiv +1 (\bmod y)$$
$$R^n \equiv -1 (\bmod z),\ \overline{R}^n \equiv -1 (\bmod z).$$

it can be shown that the coordinates x, y, z satisfy the following equation

$$0 = x^n + y^n - z^n + xyz.k(x, y, z)$$

for some integer k. This equation is termed the 'coordinate equation' in [1] and is a second conservation equation; the first is the DCE. Of course, k is not zero, (Wiles [6]), so URMT also has a link to Fermat's Last Theorem (FLT).

3) The dynamical variables and coordinates are related to each other by the following relations, for some integers α, β and γ:

$$\begin{aligned}
&(1 - P\overline{P}) = \alpha x \\
&(1 - Q\overline{Q}) = \beta y \\
&(1 - R\overline{R}) = \gamma z \\
&\alpha, \beta, \gamma \in \mathbb{Z},\ (\alpha, \beta, \gamma) \neq (0,0,0)
\end{aligned}$$

The α, β and γ are termed 'dual variables' (or scale, divisibility factors) in [1], as they are the dual forms of the coordinates x, y, z. By summing all three relations above to give

$$3 - (P\overline{P} + Q\overline{Q} + R\overline{R}) = \alpha x + \beta y + \gamma z$$

and substituting for the kinetic energy $K = P\overline{P} + Q\overline{Q} + R\overline{R}$ in terms of the potential $V = PQR + \overline{P}\,\overline{Q}\,\overline{R}$, using the DCE, $+1 = K + V$, then a third conservation equation is obtained

$$2 + V = \alpha x + \beta y + \gamma z\,.$$

This is termed the 'potential Equation' in [1].

A numeric example

Eigenvector solution

$$n = 3,\ x = 9,\ y = 31,\ z = 70,\ \mathbf{X} = \begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix}.$$

Dynamical variables P, Q, R and their conjugates $\overline{P}, \overline{Q}, \overline{R}$ for $\lambda_1 = +1$

$$\begin{aligned}
&P = -2,\ Q = -6,\ R = -11 \\
&\overline{P} = +4,\ \overline{Q} = +5,\ \overline{R} = +19\,.
\end{aligned}$$

The eigenvector equation ('dynamical equations' in [1]), in matrix form

$$\mathbf{AX} = \mathbf{X}, \begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix} = \begin{pmatrix} 0 & -11 & +5 \\ +19 & 0 & -2 \\ -6 & +4 & 0 \end{pmatrix} \begin{pmatrix} 9 \\ 31 \\ 70 \end{pmatrix}, \ \lambda_1 = +1$$

The kinetic term $K = P\overline{P} + Q\overline{Q} + R\overline{R} = -247$

The potential $V = PQR + \overline{P}\overline{Q}\overline{R} = 248$

The Dynamical Conservation Equation for $\lambda_1 = +1$

$$+1 = K + V = -247 + 248$$

Eigenvalues

$$\lambda_1 = +1$$

$$b = \frac{\sqrt{(4V-1)}}{2} = 15.74 \text{ to 2dps}$$

$$\lambda_2 = -\frac{1}{2} + 15.74 \text{ to 2dps}$$

$$\lambda_3 = -\frac{1}{2} - 15.74 \text{ to 2dps}$$

The coordinate equation $0 = x^n + y^n - z^n + xyz.k(x, y, z)$

$$0 = 9^3 + 31^3 - 70^3 + 9.31.70.k(x, y, z), \ k(x, y, z) = +16$$

For $\mathrm{Re}(\lambda_2, \lambda_3) = +1/2$, $\lambda_1 = -1$, the sign of the dynamical variables is flipped, which also flips the matrix $\mathbf{A}$ to $-\mathbf{A}$, i.e.

$P = 2$, $Q = 6$, $R = 11$
$\overline{P} = -4$, $\overline{Q} = -5$, $\overline{R} = -19$.

The same $\mathbf{X}$ is now an eigenvector -1, i.e. $(-\mathbf{A})\mathbf{X} = -\mathbf{X}$. Using $b = \left[1 + \sqrt{(-1-4V)}\right]/2$, then $b = 15.74$ to 2dps (since V also flips sign to negative, i.e. $V = -248$),

$$\lambda_2 = +\frac{1}{2} + 15.74 \text{ to 2dps}$$

$$\lambda_3 = +\frac{1}{2} - 15.74 \text{ to 2dps}$$

Now to apply a single, local variation by letting $\eta = 1$, with $\delta = 0$ and $\varepsilon = 0$ so that only the top row of the variational matrix $\mathbf{\Delta}$ is non-zero, and therefore only the top row of matrix $\mathbf{A}$ changes, that is, only the dynamical variables R and $\overline{Q}$ change as follows:

$$\eta = 1,\ \delta = 0,\ \varepsilon = 0$$
$$R \to R + \eta z \text{, i.e. } R \to 11 + 70 = 80$$
$$\overline{Q} \to \overline{Q} - \eta y \text{, i.e. } \overline{Q} \to -5 - 31 = -36 .$$

potential $V = -1764$

kinetic $K = -1763$

The DCE is verified as $-K + V = -1$ (for $\lambda_1 = -1$)

and the two complex eigenvalues are

$$\lambda_2 = +\frac{1}{2} + 41.997i \text{ to 3dps}$$

$$\lambda_3 = +\frac{1}{2} - 41.997i \text{ to 3dps}$$

Of course, the integer variation η was given the appropriate sign to make V negative and keep the eigenvalues λ_2 and λ_3 complex. It could be made such that the roots are real instead, but then this would not be the domain of the Riemann Hypothesis, albeit still URMT.

The first ten values of R, $\overline{Q}$, V and b, for η from 0 to 9, are tabulated below.

η	R	$\overline{Q}$	V	b
0	11	-5	-248	15.7401
1	81	-36	-1764	41.9970
2	151	-67	-3280	57.2691
3	221	-98	-4796	69.2514
4	291	-129	-6312	79.4465
5	361	-160	-7828	88.4746
6	431	-191	-9344	96.6631
7	501	-222	-10860	104.2101
8	571	-253	-12376	111.2463
9	641	-284	-13892	117.8633

It is seen from this example that the values for b are not the same as the actual values listed, for example, in [7]. However, the above is one very specific example for a specific exponent (cubic), and a single variation in η only. The reader may wish to find some magical combination of exponent, dynamical variables and variations that do reproduce the desired results, if at all possible - which is not currently known.

Lastly

This article is provided in good faith (without warranty!), in the hope that it may aid the progress of those seriously looking at the Riemann Hypothesis, in particular with links to physics. However, as the author, I primarily work in mathematical physics and, until now, have steered well clear of the Riemann Hypothesis as I have other interests, notably the discrete formulation of the laws of nature, i.e. URMT.

Good luck, I hope this article is of interest or use.